MATH 241 – CALCULUS III
THIRD MIDTERM EXAM SOLUTIONS

(1) Switching the order of integration, the integral is equivalent to:

$$\int_0^3 \int_0^{x^2} \sin(\pi x^3) dy dx = \int_0^3 x^2 \sin(\pi x^3) dx = -\frac{1}{3\pi} \cos(\pi x^3)\Big]_0^3 = \frac{2}{3\pi}$$

(2) The surface area is the integral of $\sqrt{1 + f_x^2 + f_y^2}$, where in this case $f(x, y) = x^2$. So

$$\int_0^1 \int_0^x \sqrt{1 + 4x^2} dy dx = \int_0^1 x\sqrt{1 + 4x^2} dx = \frac{1}{12}(1 + 4x^2)^{3/2}\Big]_0^1 = \frac{5^{3/2} - 1}{12}$$

(3) Use spherical coordinates to write the volume as

$$\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho^2 \sin\phi \, d\rho d\phi d\theta = \frac{8}{3} \int_0^{2\pi} \int_0^{\pi/4} \sin\phi \, d\phi d\theta = \frac{16\pi}{3}\left(1 - \frac{1}{\sqrt{2}}\right)$$

(4) Use the change of variables $x = 4u$, $y = 5v$. The Jacobian determinant is $4 \cdot 5 = 20$, and in the u, v-coordinates the ellipse is just the unit disk. So converting again to polar coordinates, the integral is

$$\iint_{u^2+v^2 \leq 1} 20(1 + u^2 + v^2)^{3/2} du dv = \int_0^{2\pi} \int_0^1 20(1 + r^2)^{3/2} r dr d\theta$$

$$= 40\pi \frac{(1 + r^2)^{5/2}}{5}\Bigg]_0^1 = 8\pi(2^{5/2} - 1)$$

(5) By symmetry, the sides of the rectangle are parallel to the coordinate axes, and the rectangle is centered at the origin. If (x, y) is the upper right corner, then we want to maximize the function $f(x, y) = 4xy$ subject to the constraint $g(x, y) = x^2/4 + y^2/9$. By Lagrange multipliers, we have

$$4y = \lambda x/2 \, 4x = \lambda 2y/9$$

or, $8y/x = \lambda = 18x/y$. So $4y^2 = 9x^2$, $2y = 3x$. From the constraint, this implies $x^2 = 2$, so the maximal area is $12x^2/2 = 12$.

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