MATH 241 – CALCULUS III
FOURTH MIDTERM EXAM SOLUTIONS

(1) (a) Parametrize the curve $\mathbf{r}(t) = t\,\hat{\mathbf{i}} + t^2\,\hat{\mathbf{j}}$, $0 \le t \le 1$. Then $\mathbf{F} \cdot \mathbf{r}'(t) = te^{t^2} + 2t$. Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (te^{t^2} + 2t)dt = \left[\tfrac{1}{2}e^{t^2} + t^2\right]_0^1 = \tfrac{1}{2}(e+1)$$

(b) This vanishes by Green's theorem. (c) Note that $(2xy^2+1)\,\hat{\mathbf{i}} + (2x^2y)\,\hat{\mathbf{j}} = \nabla f$, where $f(x,y) = x^2y^2 + x$. So by the Fundamental Theorem of Calculus for Line Integrals,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(2,3) - f(-1,2) = 38 - 3 = 35$$

(2) (The problem should have specified an orientation: take the upward normal) Parametrize the surface by

$$\mathbf{r}(r,\theta) = r\cos\theta\,\hat{\mathbf{i}} + 2r\sin\theta\,\hat{\mathbf{j}} + 3(1-r^2)^{1/2}\,\hat{\mathbf{k}}$$

Then

$$\begin{aligned}
\mathbf{r}_r &= \cos\theta\,\hat{\mathbf{i}} + 2\sin\theta\,\hat{\mathbf{j}} - 3r(1-r^2)^{-1/2}\,\hat{\mathbf{k}} \\
\mathbf{r}_\theta &= -r\sin\theta\,\hat{\mathbf{i}} + 2r\cos\theta\,\hat{\mathbf{j}} \\
\mathbf{r}_r \times \mathbf{r}_\theta &= \frac{6r^2\cos\theta}{(1-r^2)^{1/2}}\,\hat{\mathbf{i}} + \frac{3r^2\sin\theta}{(1-r^2)^{1/2}}\,\hat{\mathbf{j}} + 2r\,\hat{\mathbf{k}} \\
\mathbf{F} \cdot \mathbf{r}_r \times \mathbf{r}_\theta &= 18r(1-r^2)
\end{aligned}$$

So

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,dS = \int_0^{2\pi}\int_0^1 18r(1-r^2)\,dr\,d\theta = 36\pi\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 = 9\pi$$

Alternatively, notice that the surface integral of $\mathbf{F}$ on the "bottom" of the ellipsoid, where $z = 0$, is identically zero. This is because the normal vector is $-\hat{\mathbf{k}}$, and $z = 0$. Hence, we may apply the divergence theorem to get

$$\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\,dS = \int_R \int_0^{3(1-x^2-y^2/4)^{1/2}} 2z\,dz\,dx\,dy$$

Date: Dec 8, 2008.

where R is the region inside the ellipse $x^2 + (y/2)^2 = 1$. Then

$$\begin{aligned}\iint_S \mathbf{F} \cdot \hat{\mathbf{n}}\, dS &= \int_R 9(1 - x^2 - y^2/4)dxdy \\ &= \int_R 18(1 - x^2 - y^2/4)dxd(y/2) \\ &= \int_0^{2\pi} \int_0^1 18(1 - r^2)rdrd\theta \\ &= 36\pi \left[r^2/2 - r^4/4\right]_0^1 \\ &= 9\pi\end{aligned}$$

(3) (a) For a surface S with boundary ∂S,

$$\iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS = \int_{\partial S} \mathbf{F} \cdot d\mathbf{r}$$

Here, the orientation of ∂S depends on $\hat{\mathbf{n}}$ according to the right hand rule. (b)

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z(x+1) & zy - x & x \end{vmatrix} = -y\ \hat{\mathbf{i}} + x\ \hat{\mathbf{j}} -\ \hat{\mathbf{k}}$$

(c) By Stokes' Theorem we can evaluate the line integral via the surface integral over the sphere. The normal vector is given by $\hat{\mathbf{n}} = x\ \hat{\mathbf{i}} + y\ \hat{\mathbf{j}} + z\ \hat{\mathbf{k}}$. In spherical coordinates, $z = \cos\phi$. So

$$\begin{aligned}\int_{\partial S} \mathbf{F} \cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot \hat{\mathbf{n}}\, dS \\ &= \int_0^{\pi/2} \int_0^{\pi/2} -\cos\phi \sin\phi\, d\phi d\theta \\ &= -\left(\frac{\pi}{2}\right)\left[\frac{\sin^2\phi}{2}\right]_0^{\pi/2} \\ &= -\pi/4\end{aligned}$$

(4) (a) For a bounded domain D with boundary surface ∂D,

$$\iiint_D \nabla \cdot \mathbf{F}\, dV = \iint_{\partial D} \mathbf{F} \cdot \hat{\mathbf{n}}\, dS$$

Here, $\hat{\mathbf{n}}$ is the outward pointing normal. (b) $\nabla \cdot \mathbf{F} = -y + 2yz\cos(-z^2 y)$. (c) For *any* vector field, $\nabla \cdot (\nabla \times \mathbf{F}) = 0$.