Math 246H – Exam #1

1(a). By using an integrating factor, $(t^{-2}y)' = 3\ln t/t = (3(\ln t)^2/2)'$, so

$$y = \frac{3}{2}t^2(\ln t)^2 + Ct^2$$

The initial condition implies $C = 2$. The solution is defined for $t > 0$.

1(b). Separate variables to get $(y^2)' = (4\sqrt{x^2-9})'$, so

$$y = \pm(C + 4\sqrt{x^2-9})^{1/2}$$

The initial condition implies $C = -15$ and also that we must take the negative solution; so

$$y = -(4\sqrt{x^2-9} - 15)^{1/2}$$

The solution is defined for $x \geq \sqrt{9 + (15/4)^2}$.

2. The equilibrium solutions are $y = 0$, $y = 2$, and $y = \pm 3$. Checking the sign of y', 0 and 3 are stable, and 2 and -3 are unstable.

3. This is an exact equation, so solve $\psi_x = 2x + y^2$ to get $\psi = x^2 + xy^2 + f(y)$ for some function $f(y)$. Now equate

$$\psi_y = 2xy + f'(y) = 2xy + 2y$$

to get $f(y) = y^2 + C$. Hence, the solution is

$$x^2 + xy^2 + y^2 + C = 0$$

and the initial condition implies $C = -4$, so finally

$$y = \sqrt{\frac{4-x^2}{x+1}}$$

The interval of existence is $(-1, 2)$.

4. If $P(t)$ represents the remaining principal on the loan at time t, $r = 0.1$ the rate of interest, and k the annual rate of payments, then we have the initial value problem

$$P'(t) = rP(t) - k \;;\; P(0) = P_0 = 8000$$

The solution is found by using an integrating factor:

$$P(t) = \frac{k}{r} + \left(P_0 - \frac{k}{r}\right)e^{rt}$$

We find k by requiring the principal at time $t = 3$ to be zero.

$$0 = P(3) = \frac{k}{r} + \left(P_0 - \frac{k}{r}\right)e^{3t}$$

or

$$k = \frac{rP_0}{1 - e^{-3r}}$$

Note that $1 - e^{-3r} \simeq 3r$, so the payments are roughly $P_0/3$, as they should be. The total amount paid is $3k$, so the interest is

$$3k - P_0 = \frac{P_0}{1 - e^{-3r}}(3r - 1 + e^{-3r})$$

Now $3r - 1 + e^{-3r} \simeq 9r^2/2$, so the interest paid is approximately $3r^2 P_0/2 = 3(8000)/200 = 120$.