Math 246H – Exam #2

(1) The characteristic equation is $r^3 - 3r^2 + 3r - 1 = (r-1)^3$, so e^t is one solution. Plugging in $p(t)e^t$ one sees that if $p(t)$ is any quadratic polynomial we also have a solution. So the answer is $y(t) = (at^2 + bt + c)e^t$ for constants a, b, c.

(2) Guess! and notice that $y_2 = t$ is also a solution. Systematically, let $y = f(t)e^t$ for some function $f(t)$. Then

$$\begin{aligned} y &= fe^t \\ y' &= f'e^t + fe^t \\ y'' &= f''e^t + 2f'e^t + fe^t \end{aligned}$$

and so y is a solution if and only if

$$(t-1)f'' + (2t-2)f' - tf' = (t-1)f'' + (t-2)f' = 0$$

so (for $t > 1$, say)

$$\begin{aligned} \frac{d}{dt}\ln f' &= -\frac{t-2}{t-1} = -1 + \frac{1}{t-1} \\ \ln f' &= -t + \ln(t-1) \\ f' &= (t-1)e^t \end{aligned}$$

and so $f(t) = t$ up to a constant.

(3) The roots of the characteristic equation $r^2 + 2r + 2 = 0$ are $r = -1 \pm i$, so the general solution is

$$y = e^{-t}(A\cos t + B\sin t)$$

Then $y' = -y + e^{-t}(-A\sin t + B\cos t)$. Plugging in the initial conditions we have

$$1 = \frac{e^{-\pi/4}}{\sqrt{2}}(A+B)\ ,\ 4 = \frac{e^{-\pi/4}}{\sqrt{2}}(B-A)$$

so $A = -3e^{\pi/4}/\sqrt{2}$ and $B = 5e^{\pi/4}/\sqrt{2}$.

(4) (a) Independent solutions to the homogeneous equation are e^{2t} and e^{3t}, so the general solution to the inhomogeneous equation is

$$y = y_p + Ae^{2t} + Be^{3t}$$

for constants A, B. (b) Variation of parameters gives the equations

$$\begin{aligned} A'e^{2t} + B'e^{3t} &= 0 \\ 2A'e^{2t} + 3B'e^{3t} &= 0 \end{aligned}$$

where $y_p = Ae^{2t} + Be^{3t}$ for *functions* A, B. Solving these gives $-A'e^{2t} = B'e^{3t} = g$. Therefore,

$$y_p = \int_{t_0}^{t} (e^{3(t-s)} - e^{2(t-s)})g(s)ds$$

(5) The Wronskian is

$$W = \det \begin{pmatrix} 1 & 0 & 0 \\ e^{2t} & 2e^{2t} & 4e^{2t} \\ e^{-t} & -e^{-t} & e^{-t} \end{pmatrix} = 2e^{t} + 4e^{t} = 6e^{t}$$

(6) By Abel's Theorem, the Wronskian $W = W(y_1, y_2, y_3)$ satisfies

$$W(t) = W(0) \exp\left(-\int_0^t 2ds\right) = W(0)e^{-2t}$$

(the integrand "2" being the coefficient of the y'' term in the third order equation). From the initial conditions provided, $W(0) = -2$, so $W(1) = -2e^{-2}$.