Math 246H – Exam #3 Solutions

(1) We have

$$\begin{aligned}\mathcal{L}(y''') &= s\mathcal{L}(y'') - y''(0)\\ &= s(s\mathcal{L}(y') - y'(0)) - y''(0)\\ &= s(s(s\mathcal{L}(y) - y(0)) - y'(0)) - y''(0)\end{aligned}$$

Plugging in the initial values, we find

$$\mathcal{L}(y''') = s^3Y(s) - s^2 - 2s + 1$$

(2) Rewrite

$$f(t) = ((t-2)^2 + 4(t-2) + 3))u_2(t) - ((t-3) + 5)u_3(t)$$

Now from the tables,

$$\mathcal{L}(f)(s) = \left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{3}{s}\right] e^{-2s} - \left[\frac{1}{s^2} + \frac{5}{s}\right] e^{-2s}$$

(3) (a) Rewrite using partial fractions:

$$F(s) = \frac{6/5}{s-3} + \frac{4/5}{s+2}$$

Now from the tables,

$$\mathcal{L}^{-1}(F)(t) = \frac{6}{5}e^{3t} + \frac{4}{5}e^{-2t}$$

(b) Complete the square to rewrite

$$F(s) = \frac{(s-2)e^{-3s}}{(s-2)^2 + 1}$$

So, $\mathcal{L}^{-1}(F)(t) = u_3(t)e^{2t-6}\cos(t-3)$.

(4) Let $x_1 = y$, $x_2 = x_1' = y'$. Then

$$x_2' = y'' = -ty' + (t^2-1)y + \cos t = -tx_2 + (t^2-1)x_1 + \cos t$$

Therefore,

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ t^2-1 & -t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \cos t \end{pmatrix}$$

(5) The eigenvalues are 4 and -3 with eigenvectors $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$, respectively. Moreover,

$$\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{3}{7}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{2}{7}\begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

So the solution is

$$\mathbf{x}(t) = \frac{3}{7}e^{4t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{2}{7}e^{-3t}\begin{pmatrix} 2 \\ -5 \end{pmatrix}$$

(6) The eigenvalues are $1 \pm 2i$. An eigenvector for $\lambda = 1 + 2i$ is $\mathbf{v} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}$. Then

$$\begin{aligned} e^{\lambda t}\mathbf{v} &= e^t(\cos(2t) + i\sin(2t))\begin{pmatrix} 1 \\ 2i \end{pmatrix} \\ &= e^t\begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix} + ie^t\begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix} \end{aligned}$$

Therefore, the general solution is

$$\mathbf{x}(t) = Ae^t\begin{pmatrix} \cos(2t) \\ -2\sin(2t) \end{pmatrix} + Be^t\begin{pmatrix} \sin(2t) \\ 2\cos(2t) \end{pmatrix}$$

for (real) constants A and B.