# MATH 341 – EXAM # 2 SOLUTIONS

(1) Use partial fractions to show

$$\frac{s}{s^2 - 4s - 12} = \frac{1/4}{s+2} + \frac{3/4}{s-6}$$

Hence,

$$\mathcal{L}^{-1}\left(\frac{s}{s^2 - 4s - 12}\right) = \frac{1}{4}e^{-2t} + \frac{3}{4}e^{6t}$$

(2) Let $\mathcal{L}(y) = Y$. Also, the right hand side may be written as $H_{\pi/2}(t)\cos t$. Let $\Re$ denote the real part. Then

$$\begin{aligned}
\mathcal{L}(y') &= sY - y(0) = sY - 3 \\
\mathcal{L}(y'') &= s^2Y - sy(0) - y'(0) = s^2Y - 3s + 1 \\
(s^2+1)Y - 3s + 1 &= \Re \int_0^\infty H_{\pi/2}(t) e^{it} e^{-st} dt \\
&= \Re \int_0^{\pi/2} e^{it} e^{-st} dt \\
&= \Re \int_0^{\pi/2} e^{-(s-i)t} dt \\
&= \Re \left\{ -\frac{e^{-(s-i)t}}{s-i} \right]_0^{\pi/2} \Bigg\} \\
&= \Re \left\{ \frac{(1 - e^{-(s-i)\pi/2})}{s^2+1}(s+i) \right\} \\
&= \Re \left\{ \frac{(1 - ie^{-s\pi/2})}{s^2+1}(s+i) \right\} \\
&= \frac{s}{s^2+1} + \frac{e^{-s\pi/2}}{s^2+1}
\end{aligned}$$

It follows that

$$y(t) = \mathcal{L}^{-1}\left(\frac{3s-1}{s^2+1} + \frac{s + e^{-s\pi/2}}{(s^2+1)^2}\right)$$

(3) Write

$$y = \sum_{n=0}^{\infty} a_n t^n \qquad y' = \sum_{n=0}^{\infty} n a_n t^{n-1} \qquad y'' = \sum_{n=0}^{\infty} n(n-1) a_n t^{n-2}$$

Then the equation becomes

$$\sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} - 2na_n + \lambda a_n\right) t^n = 0$$

and so

$$a_{n+2} = \frac{(2n-\lambda)}{(n+1)(n+2)} a_n$$

(4) Write

$$y = \sum_{n=0}^{\infty} a_n t^{n+r} \qquad y' = \sum_{n=0}^{\infty} n a_n t^{n+r-1} \qquad y'' = \sum_{n=0}^{\infty} n(n-1)a_n t^{n+r-2}$$

Then the equation becomes

$$\sum_{n=0}^{\infty} \left((n+r)(n+r-1)a_n - (n+r)a_n - a_n\right) t^{n+r} - a_n t^{n+r+1} = 0$$

So the indicial equation is $r(r-1) + r - 1 = r^2 - 1 = 0$.

(5) Let $x_1 = y$, $x_2 = y'$, $x_3 = y''$. Then the equation is equivalent to

$$\mathbf{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & -2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} 0 \\ 0 \\ e^t \end{pmatrix}$$

with initial condition

$$\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$$

(6) Since the identity matrix commutes with everything,

$$\begin{aligned} \exp(tA) &= \exp(tI) \exp \begin{pmatrix} 0 & 0 \\ 2t & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} \left( I + \begin{pmatrix} 0 & 0 \\ 2t & 0 \end{pmatrix} \right) \\ &= \begin{pmatrix} e^t & 0 \\ 0 & e^t \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2t & 1 \end{pmatrix} = \begin{pmatrix} e^t & 0 \\ 2te^t & e^t \end{pmatrix} \end{aligned}$$

Setting $t = 1$,

$$\exp(A) = \begin{pmatrix} e & 0 \\ 2e & e \end{pmatrix}$$

The solution to the initial value problem is

$$\mathbf{x}(t) = \exp(tA)\mathbf{x}(0) = \begin{pmatrix} e^t & 0 \\ 2te^t & e^t \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^t \\ 4te^t + e^t \end{pmatrix}$$