MATH 341 – FINAL EXAM SOLUTIONS

(1) $y(t) = ae^{2t} + bte^{2t} + (2+t)e^t$, where a and b are constants.

(2) Write the equation in the form $\phi_t + \phi_y y' = 0$ for $\phi(t,y)$. Then solve

$$\phi(t,y) = ty^2 + 2t^2y^3 + 4y = \text{ constant}$$

The constant is determined by the initial condition. We find

$$ty^2 + 2t^2y^3 + 4y = -4$$

(3) (a) $\mathcal{L}(y)(s) = \int_0^\infty e^{-st}y(t)dt$. (b) Compute

$$\begin{aligned}\int_0^\infty e^{-st}\cos t dt &= \Re \int_0^\infty e^{-st}e^{it}dt \\ &= \Re \int_0^\infty e^{-(s-i)t}dt \\ &= \Re - \frac{e^{-(s-i)t}}{s-i}\Big]_0^\infty \\ &= \Re - \frac{1}{s-i} = \Re - \frac{s+i}{s^2+1} \\ &= \frac{s}{s^2+1}\end{aligned}$$

(c) $\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$, so

$$\begin{aligned}\mathcal{L}(y'') &= s\mathcal{L}(y') - y'(0) \\ \mathcal{L}(y'') &= s^2\mathcal{L}(y) - sy(0) - y'(0) \\ &= s^2\mathcal{L}(y) - s - 2\end{aligned}$$

(4) The characteristic polynomial of A is $p_A(\lambda) = -(\lambda-1)(\lambda-3)(\lambda+2)$. Use row reduction to find the eigenvectors. The general solution is

$$\mathbf{x}(t) = ae^t\begin{pmatrix}-1\\4\\1\end{pmatrix} + be^{3t}\begin{pmatrix}1\\2\\1\end{pmatrix} + ce^{-2t}\begin{pmatrix}1\\-1\\-1\end{pmatrix}$$

for constants a, b, c.

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(5) (a) At $t = 0$ (and hence, any t) the vectors are independent. (b) Choose the correct initial condition:

$$\mathbf{x}(t) = -\mathbf{x}_1(t) - \mathbf{x}_2(t) + 2\mathbf{x}_3(t)$$

(c) By the same argument choose initial conditions to be the standard basis. So the fundamental solution is

$$\mathbf{X}(t) = \begin{pmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{pmatrix} \qquad \text{so} \qquad A = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(6) (a) Solve simultaneously for $f(x,y) = g(x,y) = 0$. The solutions are

$$(0,0)\ (-3,1)\ (0,-2)\ (-4,0)$$

(b) Compute

$$f_x = y\ ,\ f_y = x + 2y + 2\ ,\ f_{xx} = 0\ ,\ f_{yy} = 2\ ,\ f_{xy} = 1$$
$$g_x = 2x - y + 4\ ,\ g_y = -x\ ,\ g_{xx} = 2\ ,\ g_{yy} = 0\ ,\ g_{xy} = -1$$

Evaluating at $(-3,1)$, we have

$$f(x,y) = (x+3) + (y-1) + (x+3)(y-1) + (y-1)^2$$
$$g(x,y) = -3(x+3) + 3(y-1) + (x+3)^2 - (x+3)(y-1)$$

(c) Linearize the equation at each of the critical points. In each case, let A be the matrix

$$A = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$$

evaluated at the critical point. Plugging in the values into the expression above, we find

$$\text{At } (0,0)\ ,\ A = \begin{pmatrix} 0 & 2 \\ 4 & 0 \end{pmatrix}$$
$$\text{At } (-3,1)\ ,\ A = \begin{pmatrix} 1 & 1 \\ -3 & 3 \end{pmatrix}$$
$$\text{At } (0,-2)\ ,\ A = \begin{pmatrix} -2 & -2 \\ 6 & 0 \end{pmatrix}$$
$$\text{At } (-4,0)\ ,\ A = \begin{pmatrix} 0 & -2 \\ -4 & 4 \end{pmatrix}$$

From the determinants and traces, we see that $(0,-2)$ is stable, and the others are all unstable.