Math 436 – Exam #1

(1) (10 pts) Let $\gamma : (a,b) \to \mathbb{R}^3$ be a smooth curve. Is it true that γ necessarily has a unit speed reparametrization? Explain.

(2) (10 pts) Let $\mathbf{U}, \mathbf{V} : \mathbb{R} \to \mathbb{R}^3$ be vector valued functions satisfying $\dot{\mathbf{U}} = \mathbf{A}\mathbf{U}$, $\dot{\mathbf{V}} = \mathbf{A}\mathbf{V}$ for a *skew-symmetric* matrix valued function $\mathbf{A}$. Show that if $\langle \mathbf{U}, \mathbf{V} \rangle$ vanishes for some t, it vanishes identically.

(3) (25 pts) Let $\gamma : [0,L] \to \mathbb{R}^2$ be a unit speed *closed* plane curve. Show that for any $t_1 \in [0,L]$ there is $t_2 \in [0,L]$ such that $\dot{\gamma}(t_1) = -\dot{\gamma}(t_2)$ (Hint: recall the expression $\dot{\gamma}(t) = (\cos\varphi(t), \sin(\varphi(t)))$.

(4) (30 pts) Consider the curve $\gamma : \mathbb{R} \to \mathbb{R}^3$ given by: $\gamma(t) = (e^t \cos t, e^t \sin t, e^t)$. Compute the curvature and torsion κ, τ, and the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$.

(5) (25 pts) Let $\gamma : (a,b) \to \mathbb{R}^3$ is a unit speed curve with $\kappa(t) > 0$ and $\tau(t) \neq 0$ for every $t \in (a,b)$. Suppose that γ lies on a sphere (of some radius R). Show that:

$$\frac{d}{dt}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right) = \frac{\tau}{\kappa}$$

(Hint: $\|\gamma - \mathbf{u}\|^2 = R^2$ for some fixed vector $\mathbf{u}$. Now differentiate!)