Math 436 – Exam #1

1. No. We need γ to be regular.

2. Skew-symmetric means $A^T = -A$. Now

$$\begin{aligned}\frac{d}{dt}\langle U, V\rangle &= \langle \dot{U}, V\rangle + \langle U, \dot{V}\rangle \\ &= \langle AU, V\rangle + \langle U, AV\rangle \\ &= \langle AU, V\rangle + \langle A^T U, V\rangle \\ &= \langle AU, V\rangle - \langle AU, V\rangle \\ &= 0\end{aligned}$$

3. We know that φ is continuous (in fact smooth) and $\varphi(t_1+L) = \varphi(t_1)+2\pi k$ for some integer k. If $k \neq 0$, then by the intermediate value theorem there is t_2 such that $\varphi(t_2) = \varphi(t_1) + \pi$ or $\varphi(t_2) = \varphi(t_1) - \pi$. This is the desired point. Unfortunately, this argument only works for $k \neq 0$! This holds, for example, for simple closed curves, but it is not true in general.

4. Compute! $ds/dt = \|\dot{\gamma}\| = e^t\sqrt{3}$, and use

$$\frac{ds}{dt}\frac{d}{ds}\mathbf{T} = \dot{\mathbf{T}}$$

etc.

$$\begin{aligned}\mathbf{T} &= \frac{1}{3}(\cos t - \sin t, \cos t + \sin t, 1) \\ \mathbf{N} &= \frac{1}{\sqrt{2}}(-\cos t - \sin t, \cos t - \sin t, 0) \\ \mathbf{B} &= \frac{1}{\sqrt{6}}(-\cos t + \sin t, -\cos t - \sin t, 2) \\ \kappa &= \sqrt{2}/3e^t \\ \tau &= 1/3e^t\end{aligned}$$

5. Differentiate $\|\gamma - \mathbf{u}\|^2 = 0$ to obtain $\langle T, \gamma - \mathbf{u}\rangle = 0$. Differentiate again and use $\|\mathbf{T}\| = 1$, $\ddot{\gamma} = \kappa\mathbf{N}$, to obtain $1 = \kappa\langle \mathbf{N}, \gamma - \mathbf{u}\rangle$. Differentiating one more time, and using $\dot{\mathbf{N}} = -\kappa\mathbf{T} + \tau\mathbf{B}$,

$\dot{\mathbf{B}} = -\tau\mathbf{N}$, we have

$$\begin{aligned} 0 &= \dot{\kappa}\langle \mathbf{N}, \gamma - \mathbf{u}\rangle + \kappa\langle \dot{\mathbf{N}}, \gamma - \mathbf{u}\rangle + \kappa\langle \mathbf{N}, \mathbf{T}\rangle \\ \frac{\dot{\kappa}}{\kappa} &= \kappa\langle \kappa\mathbf{T} - \tau\mathbf{B}, \gamma - \mathbf{u}\rangle \\ \frac{\dot{\kappa}}{\tau\kappa^2} &= -\langle \mathbf{B}, \gamma - \mathbf{u}\rangle \\ \frac{d}{dt}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right) &= -\langle \dot{\mathbf{B}}, \gamma - \mathbf{u}\rangle - \langle \mathbf{B}, \mathbf{T}\rangle \\ \frac{d}{dt}\left(\frac{\dot{\kappa}}{\tau\kappa^2}\right) &= \frac{\tau}{\kappa} \end{aligned}$$