Math 436 – Exam #2

(1) (10 pts) For what values of real numbers c does the level set $xyz^2 = c$ define a regular surface?

(2) (15 pts) Find a convenient parametrization (coordinate patch) for the hyperboloid: $x^2 + y^2 - z^2 = 1$, and compute the first fundamental form in this parametrization (hint: use z as one of the parameters).

(3) (25 pts) Let $\gamma : (a, b) \to S$ be a unit speed curve on an oriented surface with normal vector $\nu : S \to \mathbb{R}^3$.

(a) How is the second fundamental form II of S defined in terms of ν?

(b) How are the geodesic and normal curvatures, $\kappa_g(\gamma)$ and $\kappa_n(\gamma)$ of γ defined?

(c) Explain why $\kappa_n(\gamma) = \mathrm{II}(\dot{\gamma}, \dot{\gamma})$.

(4) (25 pts) Let γ, S be as in the previous problem. For $p \in S$, let $\pi_p : \mathbb{R}^3 \to T_pS$ denote the orthogonal projection.

(a) Express π_p in terms of the normal vector ν_p.

(b) Let $X(t)$ be a tangent vector field along γ (i.e. $X(t) \in T_{\gamma(t)}S$). How does one define the covariant derivative $DX/\partial t$?

(c) Let $\mathbf{x} \in \mathbb{R}^3$ be a *fixed* vector, and set $X(t) = \pi_{\gamma(t)}\mathbf{x}$. Use the above and part (iii) of Problem 3 to show that

$$\left\langle \frac{DX}{\partial t}, \dot{\gamma} \right\rangle = \langle \mathbf{x}, \nu_{\gamma(t)} \rangle \kappa_n(\gamma)$$

(5) (25 pts) Consider the surface of revolution

$$\sigma(u, v) = (f(u)\cos v, f(u)\sin v, g(u))$$

with $(f'(u))^2 + (g'(u))^2 = 1$.

(a) Compute the normal vector ν in terms of this parametrization.

(b) Fix u, and let $\gamma(t) = (f(u)\cos t, f(u)\sin t, g(u))$. Compute $\kappa_n(\gamma)$ and $\kappa_g(\gamma)$.