Math 436 – Exam #2 Solutions

1. If $f(x,y,z) = xyz^2$, then $\nabla f = (yz^2, xz^2, 2xyz)$, and this vanishes only if either $z = 0$ or $(x,y) = (0,0)$. If $c \neq 0$, neither possibility holds, and so by the implicit function theorem the surface $f = c$ is regular.

2. Choose coordinates

$$\sigma(\theta, z) = \big((1+z^2)^{1/2}\cos\theta, (1+z^2)^{1/2}\sin\theta, z\big)$$

Then

$$\begin{aligned}
\sigma_\theta &= \big(-(1+z^2)^{1/2}\sin\theta, (1+z^2)^{1/2}\cos\theta, 0\big) \\
\sigma_z &= \big(z(1+z^2)^{-1/2}\cos\theta, z(1+z^2)^{-1/2}\sin\theta, 1\big) \\
E &= \|\sigma_\theta\|^2 = 1+z^2 \\
F &= \langle \sigma_\theta, \sigma_z \rangle = 0 \\
G &= \|\sigma_z\|^2 = \frac{1+2z^2}{1+z^2}
\end{aligned}$$

3. (a) For tangent vectors X, Y, $\mathrm{II}(X,Y) = -\langle D\nu(X), Y\rangle$. (b) We have an orthogonal decomposition

$$\ddot\gamma = \kappa_g(\gamma)\nu \times \dot\gamma + \kappa_n(\gamma)\nu$$

where

$$\kappa_g(\gamma) = \langle \ddot\gamma, \nu \times \dot\gamma\rangle\ ,\ \kappa_n(\gamma) = \langle \ddot\gamma, \nu\rangle$$

(c) By differentiating, we get

$$\begin{aligned}
0 &= \langle \dot\gamma, \nu(\gamma)\rangle \\
0 &= \langle \ddot\gamma, \nu(\gamma)\rangle + \langle \dot\gamma, D\nu(\gamma)\dot\gamma\rangle \\
\mathrm{II}(\dot\gamma, \dot\gamma) &= \kappa_n(\gamma)
\end{aligned}$$

4. (a) $\Pi_p(X) = X - \langle X, \nu(p)\rangle\nu(p)$. (b) $DX/\partial t = \Pi_{\gamma(t)}\dot X$. For (c), note that

$$\begin{aligned}
X(t) = \Pi_{\gamma(t)} &= \mathbf{x} - \langle \mathbf{x}, \nu(\gamma(t))\rangle\nu(\gamma(t)) \\
\dot X &= -\langle \mathbf{x}, D\nu(\gamma)\dot\gamma\rangle\nu - \langle \mathbf{x}, \nu(\gamma)\rangle D\nu(\gamma)\dot\gamma \\
\Pi_\gamma \dot X &= -\langle \mathbf{x}, \nu(\gamma)\rangle D\nu(\gamma)\dot\gamma
\end{aligned}$$

and so

$$\begin{aligned}
\left\langle \frac{DX}{\partial t}, \dot\gamma \right\rangle &= -\langle \mathbf{x}, \nu(\gamma)\rangle\langle D\nu(\gamma)\dot\gamma, \dot\gamma\rangle \\
&= \langle \mathbf{x}, \nu(\gamma)\rangle\, \mathrm{II}(\dot\gamma, \dot\gamma) \\
&= \langle \mathbf{x}, \nu(\gamma)\rangle\kappa_n(\gamma)
\end{aligned}$$

5. (a) We have

$$\begin{aligned}
\sigma_u &= (f'(u)\cos v, f'(u)\sin v, g'(u)) \\
\sigma_v &= (-f(u)\sin v, f(u)\cos v, 0) \\
\sigma_u \times \sigma_v &= (f(u)g'(u)\cos v, -f(u)g'(u)\sin v, f(u)f'(u)) \\
\nu &= \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|} = (-g'(u)\cos v, -g'(u)\sin v, f'(u))
\end{aligned}$$

(b) We have

$$\begin{aligned}
\dot\gamma &= (-f(u)\sin t, f(u)\cos t, 0) \\
T &= (-\sin t, \cos t, 0) \\
\frac{dT}{ds} &= \frac{1}{ds/dt}\dot T = \frac{1}{f}(-\cos t, -\sin t, 0) \\
\kappa_n(\gamma) &= \langle dT/ds, \nu\rangle = \frac{g'(u)}{f(u)} \\
\nu \times T &= (-f'(u)\cos t, -f'(u)\sin t, -g'(u)) \\
\kappa_g(\gamma) &= \langle dT/ds, \nu \times T\rangle = \frac{f'(u)}{f(u)}
\end{aligned}$$