Math 436 – Exam #3 : COMPLETE ANY FOUR OF THE FOLLOWING FIVE PROBLEMS.

(1) Let S be the graph $z = f(x, y)$, where $f(x, y) = Ax^2 + Bxy + Cy^2$ for some constants A, B, C. Compute the Gauss curvature K at the point $(0, 0, 0)$.

(2) The *mean curvature* H gets its name from the following: Given $p \in S$ let $\mathbf{e}_1, \mathbf{e}_2 \in T_pS$ denote the directions of principal curvature, and let $\kappa_n(\theta)$ denote the normal curvature of a curve through p whose tangent vector makes an angle θ with $\mathbf{e}_1$ (recall that the normal curvature of a curve γ in terms of the second fundamental form is $\mathrm{II}(\dot{\gamma}, \dot{\gamma})$). Show that:

$$H(p) = \frac{1}{2\pi} \int_0^{2\pi} \kappa_n(\theta)\, d\theta$$

(3) Let $\sigma(u, v)$ be a coordinate patch for a regular surface S, and let N denote the normal vector as a vector valued function of (u, v). Then the derivative N_u is a tangent vector, and so can be expressed: $N_u = A\sigma_u + B\sigma_v$. Compute A and B in terms of the entries E, F, G and l, m, n of the first and second fundamental forms.

(4) Let $\gamma : (-1, 1) \to \mathbb{R}^3$ be a unit speed curve with nonzero curvature and binormal $\mathbf{B}(t)$. Consider

$$\sigma(t, s) = \gamma(t) + s\mathbf{B}(t)$$

(a) Show that there is $\varepsilon > 0$ so that σ defines a regular surface S for $(t, s) \in (-\varepsilon, \varepsilon) \times (-\varepsilon, \varepsilon)$.

(b) Show that $\gamma(t)$, $t \in (-\varepsilon, \varepsilon)$ is a geodesic in S.

(5) Recall that the equations for a geodesic in a parametrization are

$$\frac{d}{dt}(E\dot{u} + F\dot{v}) = \frac{1}{2}(E_u\dot{u}^2 + 2F_u\dot{u}\dot{v} + G_u\dot{v}^2)$$
$$\frac{d}{dt}(F\dot{u} + G\dot{v}) = \frac{1}{2}(E_v\dot{u}^2 + 2F_v\dot{u}\dot{v} + G_v\dot{v}^2)$$

where E, F, G are the components of the first fundamental form. Consider the parametrization of the helix S:

$$\sigma(u, v) = (u \cos v, u \sin v, v)$$

and let $\gamma(t) = \sigma(u(t), v(t))$ be a curve on S.

- Show that γ is unit speed if and only if

$$\dot{u}^2 + (1 + u^2)\dot{v}^2 = 1$$

- Show that if γ is a geodesic, then $\dot{v} = \dfrac{\Omega}{1 + u^2}$, for some constant Ω.
- For γ a geodesic, derive an equation for v as a function of u (i.e. write down an expression for dv/du; you do not have to evaluate the integral.