Math 436 – Exam #3 Solutions

(1) Choose coordinates $\sigma(x,y) = (x, y, f(x,y))$. At the point $(0,0,0)$, we have:

$$\sigma_x = (1,0,0) \ , \ \sigma_y = (0,1,0) \ , \ \mathbf{N} = (0,0,1)$$

$$\sigma_{xx} = (0,0,2A) \ , \ \sigma_{yy} = (0,0,2C) \ , \ \sigma_{xy} = (0,0,B)$$

It follows that $E = 1$, $F = 0$, $G = 1$, and $l = \langle \mathbf{N}, \sigma_{xx} \rangle = 2A$, $m = \langle \mathbf{N}, \sigma_{xy} \rangle = B$, $n = \langle \mathbf{N}, \sigma_{nn} \rangle = 2C$. Hence,

$$K = \frac{ln - m^2}{EG - F^2} = 4AC - B^2$$

(2) Let κ_1, κ_2 be the principal curvatures at p. We have $\dot{\gamma} = \cos\theta\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2$. Hence,

$$\begin{aligned}
\kappa_n(\theta) &= \mathrm{II}(\dot{\gamma}, \dot{\gamma}) \\
&= \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \\
&= \frac{1}{2}(\kappa_1 + \kappa_2) + \frac{1}{2}(\kappa_1 - \kappa_2)\cos(2\theta) \\
&= H(p) + \frac{1}{2}(\kappa_1 - \kappa_2)\cos(2\theta)
\end{aligned}$$

The result now follows by integrating from 0 to 2π.

(3) We have

$$\begin{aligned}
-l &= \langle \mathbf{N}, \sigma_{uu} \rangle = \langle \mathbf{N}_u, \sigma_u \rangle = AE + BF \\
-m &= \langle \mathbf{N}, \sigma_{uv} \rangle = \langle \mathbf{N}_u, \sigma_v \rangle = AF + BG
\end{aligned}$$

In matrix form

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} = - \begin{pmatrix} l \\ m \end{pmatrix}$$

Inverting the matrix, we have

$$A = \frac{mF - lG}{EG - F^2} \ , \ B = \frac{lF - mE}{EG - F^2}$$

(4) For part (a), notice that $\sigma_t = \dot{\gamma} + s\dot{\mathbf{B}}$, $\sigma_s = \mathbf{B}$ are independent at $(t,s) = (0,0)$. The result follows from the implicit function theorem. For part (b), $\ddot{\gamma} = \kappa \mathbf{N}$, which is orthogonal to $\sigma_t(t,0)$ and $\sigma_s(t,0)$. Hence, γ is a geodesic.

(5) We have

$$\sigma_u = (\cos v, \sin v, 0) \ , \ \sigma_v = (-u\sin v, u\cos v, 1)$$

Compute: $E = 1$, $F = 0$, $G = 1 + u^2$. Then the first two statements follow immediately. Use the first statement to derive

$$\dot{u} = 1 - \frac{\Omega^2}{1 + u^2} = \frac{1 - \Omega^2 + u^2}{1 + u^2}$$

(or its negative; note also that $\Omega^2 \leq 1$. Then

$$\frac{dv}{du} = \frac{\dot{v}}{\dot{u}} = \frac{\Omega}{1 - \Omega^2 + u^2}$$