**Math 436 – Final Exam**

(1) Write down the Frenet-Serret formulas for a unit speed curve $\gamma : (a, b) \to \mathbb{R}^3$ with nonzero curvature. Explain the meaning of each quantity in your expressions.

(2) Answer each of the following "true" or "false." Explain your answer, and if "false" provide a counterexample.

(a) Let $S \subset \mathbb{R}^3$ be a regular surface. Then between any two points in $S$ there exists a *unique* geodesic.

(b) The unit sphere $S^2 \subset \mathbb{R}^3$ is locally conformal to the plane $\mathbb{R}^2$.

(c) There exists a simple, closed, convex curve $\gamma(t) \in \mathbb{R}^2$, with arc length parameter $0 \leq t \leq \pi$, such that the curvature function is $\kappa(t) = \cos^2 t$.

(d) Let $S_1, S_2 \subset \mathbb{R}^3$ be two oriented surfaces and suppose there is an isometry $f : S_1 \to S_2$. Then $S_1$ and $S_2$ have the same principal curvatures.

(3) (a) We can express tangent vector fields $X$, $Y$ in terms of $\sigma_u$ and $\sigma_v$. With respect to this expression write down a formula for the directional derivative $D_X Y$.

(b) Using the above expression, write down the conditions for the covariant derivative $\nabla_X Y = 0$.

(4) Let $S \subset \mathbb{R}^3$ be an oriented surface with normal vector $\mathbf{N}$. Give complete definitions of the following:

(a) The first fundamental form.

(b) The second fundamental form.

(c) The principal curvatures.

(d) The mean curvature.

(e) The Gauss curvature.

(5) (a) What is the Euler characteristic of the surface obtained by removing three disjoint disks from the sphere $S^2$ (a "pair of pants")?

(b) State the Gauss-Bonnet theorem for a closed oriented surface $S \subset \mathbb{R}^3$.

(c) State the Gauss-Bonnet theorem for a triangle in $S$ where the edges are all geodesics and the *interior* angles are $\alpha$, $\beta$, $\gamma$.