Math 436 – Final Exam Solutions

(1) Let $\mathbf{T} = \dot{\gamma}$. By assumption, $\dot{\mathbf{T}} \neq 0$. Then the Frenet-Serret formula is

$$\begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{pmatrix}$$

where κ, τ are the curvature and torsion, and the normal and binormal are defined by

$$\mathbf{N} = \frac{\dot{\mathbf{T}}}{\|\dot{\mathbf{T}}\|}\ , \quad \mathbf{B} = \mathbf{T} \times \mathbf{N}$$

(2) (a) False. Any longitude between the north and south poles of a sphere is a geodesic.
(b) True. We proved this using stereographic projection.
(c) False. κ would have only three critical points, violating the four vertex theorem.
(d) False. The cylinder of radius R and plane are locally isometric, but the cylinder has principal curvatures 0 and $1/R$.

(3) (a) If $X = X_1\sigma_u + X_2\sigma_v$, and $Y = Y_1\sigma_u + Y_2\sigma_v$, for functions X_i, Y_i of u, v, then

$$D_X Y = X_1(Y_1\sigma_u + Y_2\sigma_v)_u + X_2(Y_1\sigma_u + Y_2\sigma_v)_v$$

(b) The covariant derivative vanishes if $D_X Y$ is normal to the surface. Hence, the equations are

$$0 = \langle \sigma_u, X_1(Y_1\sigma_u + Y_2\sigma_v)_u + X_2(Y_1\sigma_u + Y_2\sigma_v)_v \rangle$$
$$0 = \langle \sigma_v, X_1(Y_1\sigma_u + Y_2\sigma_v)_u + X_2(Y_1\sigma_u + Y_2\sigma_v)_v \rangle$$

(4) Let X, Y denote tangent vectors. Then
(a) $\mathrm{I}(X, Y) = \langle X, Y \rangle$.
(b) $\mathrm{II}(X, Y) = -\langle D_X \mathbf{N}, Y \rangle$.
(c) Since II is symmetric it can be diagonalized. The eigenvalues $\kappa_1 \leq \kappa_2$ are the principal curvatures.
(d) $H = (\kappa_1 + \kappa_2)/2$.
(e) $K = \kappa_1\kappa_2$.

(5) (a) It is just the Euler characteristic of the sphere minus 3 vertices; so -1.
(b) $\int_S K dA = 2\pi\chi(S)$.
(c) $\int_\Delta K dA = \alpha + \beta + \gamma - \pi$.