MATH 436 – HOMEWORK #4 – DUE OCT 25

Let $S \subset \mathbb{R}^3$ be a regular surface, and $p \in S$.

(1) Show that we may find coordinates $\sigma(u, v)$, $\sigma(0, 0) = p$ such that the first fundamental form:

$$I = E(u, v)du^2 + 2F(u, v)dudv + G(u, v)dv^2$$

satisfies $E(0, 0) = G(0, 0) = 1$ and $F(0, 0) = 0$.

(2) Suppose S is given by the coodinate

$$\sigma(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2\right)$$

Compute the first and second fundamental forms.

(3) Let $\varphi : S \to \mathbb{R}$ be a smooth function and $\sigma(u, v)$ a coordinate. Let $\Phi = \varphi \circ \sigma$ be the composition, and Φ_u, Φ_v, the partial derivatives. The *gradient* of φ at p is the unique vector $\nabla f(p) \in T_pS$ satisfying

$$\langle \nabla\varphi(p), X\rangle = D\varphi(p)(X)$$

for all $X \in T_pS$.

(a) Show that in local coordinates:

$$\nabla\varphi(p) = \frac{\Phi_u G - \Phi_v F}{EG - F^2}\sigma_u + \frac{\Phi_v E - \Phi_u F}{EG - F^2}\sigma_v$$

(b) Suppose $\nabla\varphi(p) \neq 0$. Show that $D\varphi(p)(X)$ is maximized among all unit vectors $X \in T_pS$ if and only if

$$X = \frac{\nabla\varphi(p)}{\|\nabla\varphi(p)\|}$$