Math 660 – Exam #1

(1) (i) Suppose f is an entire function, d is an integer, and there is a constant M such that

$$\int_{|z|=r} |f(z)||dz| \leq Mr^d$$

for all $r \geq 0$. What can you say about f?

Solution: It is a polynomial of degree $\leq d-1$. Indeed, from Cauchy's formula, for any z and $R > |z|$,

$$\begin{aligned} |f^{(d)}(z)| &= \left| \frac{d!}{2\pi i} \int_{|w|=R} \frac{f(w)dw}{(w-z)^{d+1}} \right| \\ &\leq \frac{d!}{2\pi} \int_{|w|=R} \frac{|f(w)||dw|}{|w-z|^{d+1}} \\ &\leq \frac{d!}{2\pi} \frac{MR^d}{(R-|z|)^{d+1}} \end{aligned}$$

Since R is arbitrary and the right hand side above vanishes as $R \to +\infty$, $f^{(d)}(z)$ vanishes identically.

(ii) Suppose f is an entire function such that $|f(z)| \geq 1$ for all z. What can you say about f?

Solution: f is constant. Indeed, $1/f(z)$ is entire and bounded by 1; hence, constant by Liouville's theorem.

(2) What does it mean for a function $f : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ to be $\mathbb{C}$-differentiable? What is the relation with the Cauchy-Riemann equations? At which points is the function $f(x, y) = (x^2 + y^2, x^2 - y^2)$ $\mathbb{C}$-differentiable?

Solution: $f = u + iv$ is $\mathbb{C}$-differentiable at z if it is differentiable as a map $\mathbb{R}^2 \to \mathbb{R}^2$ and $Df(z)J = JDf(z)$, where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This is equivalent to saying that the Cauchy-Riemann equations: $u_x = v_y$, $u_y = -v_x$, are satisfied at z. It is also equivalent to the existence of the limit

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{C}}} \frac{f(z+h) - f(z)}{h}$$

In the example given, both CR equations imply $x = -y$, so f is $\mathbb{C}$-differentiable exactly there.

(3) Compute the Laurent series expansion in *all possible* annuli of the function

$$f(z) = \frac{1}{z(z-1)(z-3)}$$

Solution: Write

$$f(z) = \frac{1}{2z}\left(\frac{-1}{z-1} + \frac{1}{z-3}\right) = \frac{1}{3z} - \frac{1}{2(z-1)} + \frac{1}{6(z-3)}$$

Then

- for $0 < |z| < 1$,
$$f(z) = \frac{1}{2z}\left(\sum_{k\geq 0} z^k - \frac{1}{3}\sum_{k\geq 0}\left(\frac{z}{3}\right)^k\right)$$
- for $1 < |z| < 3$,
$$f(z) = \frac{1}{2z}\left(-\frac{1}{z}\sum_{k\geq 0}\left(\frac{1}{z}\right)^k - \frac{1}{3}\sum_{k\geq 0}\left(\frac{z}{3}\right)^k\right)$$
- for $3 < |z| < +\infty$,
$$f(z) = \frac{1}{2z}\left(-\frac{1}{z}\sum_{k\geq 0}\left(\frac{1}{z}\right)^k + \frac{1}{z}\sum_{k\geq 0}\left(\frac{3}{z}\right)^k\right)$$

(4) (i) Compute the radius of convergence about the origin of the series $\displaystyle\sum_{k=1}^{\infty}\frac{z^{k^2}}{k^4}$.

Solution: Let $a_n = 1/n^2$ for $n = k^2$ for some $k = 1, 2, \ldots$, and $a_n = 0$ otherwise. Then $\limsup_{n\to\infty}(a_n)^{1/n} = 1$, so the radius of convergence $= 1$ by Abel's theorem (note that limit doesn't exist, but the lim sup does).

(ii) Does there exist an analytic function f in a neighborhood of the origin such that $f^{(k)}(0) \geq k!k^k$? Explain.

Solution: No. Since $a_k = f^{(k)}(0)/k!$ and $\limsup(a_k)^{1/k} = +\infty$, the radius of convergence would be $= 0$, which is a contradiction.

(5) State Morera's theorem. Give a complete proof of the following statement: If $\{f_j\}$ is a sequence of holomorphic functions on an open set $U \subset \mathbb{C}$, and if $f_j \to f$ uniformly on compact subsets of U, then f is a holomorphic function.

Solution: Morera's theorem: If $f : U \subset \mathbb{C} \to \mathbb{C}$ is continuous and

$$\int_\gamma f(z)dz = 0$$

for all closed curves $\gamma \subset U$, then f is holomorphic. For the second part, notice that by Cauchy's theorem and the fundamental theorem of Calculus applied, for example, to rectangles, a holomorphic function on a disk $D \subset\subset U$ has a primitive. Therefore, the integral of a holomorphic function around *any* closed curve in D vanishes. With this understood, note that uniform convergence $f_j \to f$ implies f is continuous. By the above discussion, if γ is a curve in D, then $\displaystyle\int_\gamma f_j(z)dz = 0$ for all j. But then by uniform convergence

$$\int_\gamma f(z)dz = \lim \int_\gamma f_j(z)dz = 0$$

for all $\gamma \subset D$. By Morera's theorem, f is then holomorphic on D. Since D was arbitrary, f is holomorphic on U.