Math 660 – Exam #2

(1) Let $n \geq 1$ be an integer and $\lambda \in \mathbb{C}$, $0 < |\lambda| < 1$. Set

$$f(z) = (z-1)^n e^z - \lambda$$

(a) Show that f has n zeros in the region $|z-1| < 1$.
(b) What can you say about the multiplicities of the zeros in part (a)?
(c) Show that there are no other zeros of f (Hint: what if $\lambda \to 0$?)

Solution: (a) Apply Rouché's theorem to $g(z) = (z-1)^n e^z$. (b) Compute:

$$f'(z) = (z-1)^n e^z \left(1 + \frac{n}{z-1}\right)$$

At the zeros of f in question, $f'(z) = \lambda(1+n/(z-1)) \neq 0$, since $|z-1| < 1$. Hence, the multiplicities are all $= 1$ (i.e. simple zeros). (c) For $\lambda = 0$ the corresponding function has a zero at $z = 1$ with multiplicity n. Now apply continuity and the argument principle to conclude that f has n zeros as well. Alternatively, by continuity and since all the zeros are at $z = 1$ for $\lambda = 0$, if (c) doesn't hold there is some $0 < \lambda < 1$ for which f has a zero with $|z-1| = 1$. But then $|(z-1)^n e^z| \geq 1$; contradiction.

(2) Let $U \subset \mathbb{C}$ be a domain. Let z_1, z_2 be in the same component of the complement of U.

(a) Show that there is a holomorphic function f on U such that

$$f'(z) = \frac{1}{z-z_1} - \frac{1}{z-z_2}$$

(b) Show that there is a holomorphic function g on U such that

$$e^{g(z)} = \frac{z-z_1}{z-z_2}$$

Solution: (a) Since z_1 and z_2 are in the same component of U^c, for any closed curve γ in U $W(\gamma, z_1) = W(\gamma, z_2)$. Hence,

$$\frac{1}{2\pi i} \int_\gamma dz \left(\frac{1}{z-z_1} - \frac{1}{z-z_2}\right) = W(\gamma, z_1) - W(\gamma, z_2) = 0$$

In particular, the integrand has a primitive in U. (b) Let f be the primitive in part (a). Then

$$\frac{d}{dz}\left(e^f \frac{(z-z_2)}{(z-z_1)}\right) = 0$$

The result follows.

(3) Let $\mathbb{D}$ be the unit disk, and $f : \mathbb{D} \to \mathbb{C}$ a holomorphic function. Suppose that $f(\mathbb{D}) \subset \{w : \mathrm{Re}\, w > 0\}$, and $f(0) = 1$. Show that for all $z \in \mathbb{D}$,

$$|f(z)| \leq \frac{1+|z|}{1-|z|}$$

Solution: $g(z) = (z-1)/(z+1)$ maps the right half conformally onto the disk. Hence, $g \circ f(z)$ maps the disk to itself and satisfies $g \circ f(0) = 0$. By the Schwarz lemma, $|g \circ f(z)| \leq |z|$, or

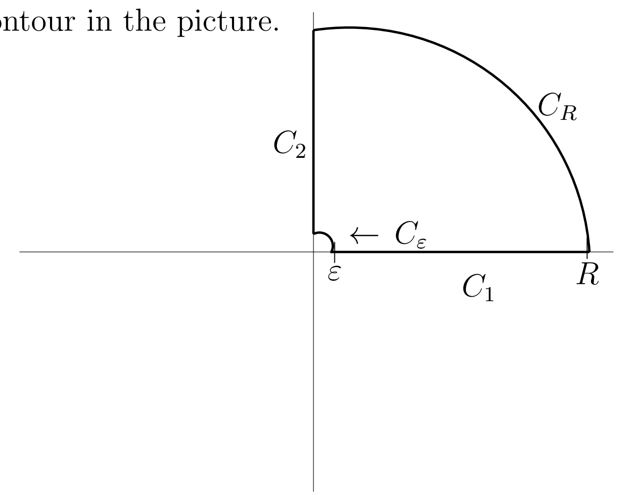
$$\begin{aligned} \left|\frac{f(z)-1}{f(z)+1}\right| &\leq |z| \\ |f(z)-1| &\leq |z||f(z)+1| \\ |f(z)|-1 &\leq |z|(|f(z)|+1) \\ |f(z)| &\leq \frac{1+|z|}{1-|z|} \end{aligned}$$

(4) For $0 < \alpha < 1$, show that

$$\int_0^\infty dx\, x^{-\alpha} \cos x = \sin(\pi\alpha/2)\Gamma(1-\alpha)$$

In particular, justify the convergence of the improper integral (Hint: convert to a complex integral over an appropriate closed contour in the upper right quadrant).

Solution: Choose the contour in the picture.



Then show that the integrals over C_ε and C_R vanish as $\varepsilon \to 0$ and $R \to \infty$. Using the principal branch of the logarithm to define powers we have $(it)^{-\alpha} = t^{-\alpha} e^{-\alpha\pi i/2}$.

So by the residue theorem (there are no poles in the interior of the contour)

$$\begin{aligned}\lim_{R\to\infty}\int_{C_1} dz\, z^{-\alpha}e^{iz} &= \lim_{R\to\infty}\int_{C_2} dz\, z^{-\alpha}e^{iz}\\ &= \lim_{R\to\infty}\int_0^R idt\, t^{-\alpha}e^{-t}e^{-\alpha\pi i/2}\\ &= ie^{-\alpha\pi i/2}\Gamma(1-\alpha)\end{aligned}$$

Taking real parts gives the result.

(5) Let $U \subset \mathbb{C}$ be a domain. For each compact set $K \subset U$, show that there is a constant $C_K \geq 1$ (depending on K) such that for all positive harmonic functions u on U and all $z, w \in K$,

$$\frac{1}{C_K} \leq \frac{u(z)}{u(w)} \leq C_K$$

Solution: Since we can rescale, it suffices to prove the result for functions in

$$\mathcal{F} = \{u \text{ positive harmonic on } U \text{ and } \max_{z\in K} u(z) = 1\}$$

I claim there is $c > 0$ such that $u(z) \geq c_K$ for all $u \in \mathcal{F}$ and all $z \in K$. This proves the result, because then

$$c_K \leq u(z)/u(w) \leq 1/c_K$$

If such a constant does not exist, there is a sequence of functions $u_j \in \mathcal{F}$ and points $z_j \in K$. By (a consequence of) the Harnack inequality, we may assume $u_j \to u$ uniformly on K, where u is harmonic and $\max_{z\in K} u(z) = 1$. But (after passing to a subsequence) $z_j \to z \in K$. Then $u(z) = 0$. By the minimum principle, u must be constant, which is a contradiction.