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Convergence properties of the Yang-Mills flow on Kähler surfaces

Dedicated to Professor Karen K. Uhlenbeck, on the occasion of her 60th birthday

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Abstract. Let E be a hermitian complex vector bundle over a compact Kähler surface X with Kähler form ω, and let D be an integrable unitary connection on E defining a holomorphic structure D'' on E. We prove that the Yang-Mills flow on (X, ω) with initial condition D converges, in an appropriate sense which takes into account bubbling phenomena, to the double dual of the graded sheaf associated to the ω-Harder-Narasimhan-Seshadri filtration of the holomorphic bundle (E, D''). This generalizes to Kähler surfaces the known result on Riemann surfaces and proves, in this case, a conjecture of Bando and Siu.

1. Introduction

The main concern of this paper is the relationship between the Yang-Mills functional and stability of holomorphic vector bundles on Kähler surfaces. By analogy with finite dimensional symplectic geometry on the one hand, and geometric invariant theory on the other, minimizing solutions to the Yang-Mills equations can be viewed as zeros of a moment map on an infinite dimensional symplectic manifold, and the appearance of the stability condition for the existence of such solutions may be regarded as a version of the Kempf-Ness Theorem. More generally, one might expect a correspondence between two stratifications: the stable-unstable manifolds defined by the gradient flow of the Yang-Mills functional, and the algebraic stratification coming from the maximally destabilizing one parameter subgroups.

This point of view originated in the work of Atiyah and Bott [AB] and was developed further by Donaldson [Do1], [Do2] (see [Ki] for a general treatment of symplectic geometry

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vs. geometric invariant theory in finite dimensions). Let us recall some of the key points. Given a holomorphic structure $\bar{\partial}_E$ on a complex vector bundle E of rank R over a Kähler manifold X, one can associate a filtration by holomorphic subsheaves, called the Harder-Narasimhan filtration, whose successive quotients are semistable. The topological type of the pieces in the associated graded object is encoded into an R-tuple $\vec{\mu} = (\mu_1, \dots, \mu_R)$ of rational numbers called the Harder-Narasimhan type of $(E, \bar{\partial}_E)$. Atiyah and Bott used the Harder-Narasimhan type to define a stratification of the infinite dimensional space $\mathscr{A}''$ of holomorphic structures on E. The group $\mathfrak{G}^{\mathbb{C}}$ of complex automorphisms of E, or the complex gauge group, acts on $\mathscr{A}''$ in a manner that preserves the stratification. The main result of [AB] is that, when X is a Riemann surface, the stratification obtained from the Harder-Narasimhan filtration is $\mathfrak{G}^{\mathbb{C}}$-equivariantly perfect, and this leads to a recursive calculation of the cohomology of the moduli spaces of stable bundles, in certain cases.

On the other hand, fixing a hermitian structure H on E, one may identify $\mathscr{A}''$ with the space $\mathscr{A}_H$ of unitary connections on E via the map which sends a unitary connection D to its $(0,1)$ part $D'' = \bar{\partial}_E$ (in higher dimensions we require the integrability condition that the curvature F_D be of type $(1,1)$). The Yang-Mills functional, which associates to a connection the L^2-norm of its curvature, can be used as a Morse function on $\mathscr{A}_H$. For a fixed holomorphic bundle $(E, \bar{\partial}_E)$, the Harder-Narasimhan type gives an absolute lower bound on the Yang-Mills number of any connection in the isomorphism class. Up to a topological term, the Yang-Mills number is the same as the Hermitian-Yang-Mills number, which is defined as the L^2 norm of the contraction ΛF_D of the curvature with the Kähler form (see (2.4)). If $\vec{\mu}$ is the Harder-Narasimhan type of $(E, \bar{\partial}_E)$, then:

$$\mathrm{HYM}(\vec{\mu}) := 2\pi \sum_{i=1}^{R} \mu_i^2 \leqq \mathrm{HYM}(D) := \int_X |\Lambda F_D|^2 \, dvol$$

for all D such that (E, D'') is holomorphically isomorphic to $(E, \bar{\partial}_E)$ (see Cor. 2.22 for a proof of this result for Kähler surfaces). Atiyah and Bott conjectured that, on a Riemann surface, the gradient flow of the Yang-Mills functional should converge at infinity, achieving the lower bound expressed above. Moreover, the stable-unstable manifold stratification should coincide with the Harder-Narasimhan stratification.

That this is indeed the case follows from the work of several authors. First, Donaldson proved the long time existence of the L^2-gradient flow for the Yang-Mills functional on any Kähler manifold [Do1], [DoKr]. For Riemann surfaces, the asymptotic convergence of the gradient flow and the equivalence of the two stratifications was established in [D] (see also [R] for a more analytic approach). A key fact which makes the two dimensional case more tractable is that the Yang-Mills functional on Riemann surfaces satisfies the equivariant Palais-Smale Condition C.

In higher dimensions Condition C fails. More seriously, the Yang-Mills flow can develop singularities in finite time. Therefore, one cannot expect a Morse theory in the classical sense. For Riemannian four-manifolds, Taubes formulated an extended Morse theory by attaching bundles with varying topologies in order to compensate for curvature concentration, or *bubbling*. In this approach one uses the strong gradient of the Yang-Mills functional associated to a complete Riemannian metric on the space $\mathscr{A}_H$, which exists for all time by the fundamental existence theorem for ODE's. In this way Taubes established the connectivity of the moduli space of self dual connections in certain cases. Moreover, he was able to calculate the stable homotopy groups as predicted by the conjecture of Atiyah and Jones [T].

In the case of Kähler surfaces it is more natural to consider the L^2 rather than the strong gradient flow. Long time existence, as mentioned above, is guaranteed. Donaldson used this flow to prove the correspondence between anti-self-dual connections and stable bundles (see [Do1], and more generally [Do2] and [UY] for holomorphic bundles in higher dimensions). This is now known as the Hitchin-Kobayashi Correspondence, or the Donaldson-Uhlenbeck-Yau Theorem. In [BS], Bando and Siu extended the correspondence to coherent analytic sheaves by considering singular hermitian metrics with controlled curvature. They also conjectured that the relationship between the Yang-Mills flow and the Harder-Narasimhan filtration which holds on Riemann surfaces should analogously be true in higher dimensions. We use the word "analogous" because even when considering vector bundles the Harder-Narasimhan filtration in higher dimensions may be given only by subsheaves, and the associated graded objects may not be locally free. Indeed, it is for this reason that the generalization of Bando-Siu naturally leads to a conjecture on the behavior of the flow for vector bundles.

The purpose of this paper is to prove the conjecture of Bando and Siu for holomorphic bundles on Kähler surfaces. To state the result precisely, let $\mathrm{Gr}_{\omega}^{\mathrm{hns}}(E, \bar{\partial}_E)$ denote the Harder-Narasimhan-Seshadri filtration of $(E, \bar{\partial}_E)$ with respect to the Kähler form ω, and let $\mathrm{Gr}_{\omega}^{\mathrm{hns}}(E, \bar{\partial}_E)^{**}$ be its double dual. To clarify, we note here that the Harder-Narasimhan-Seshadri filtration is actually a double filtration which takes into account the possibility that the successive factors in the Harder-Narasimhan filtration may only be semistable as opposed to stable (see Prop. 2.6). Thus, the individual factors in the associated graded object are all stable. Since X is now assumed to be a surface, the double dual is a vector bundle and carries a Yang-Mills connection which realizes the hermitian structure as a direct sum of Hermitian-Einstein metrics. Now for unstable bundles the flow may not converge in the usual sense; again, because of bubbling. However, one can always extract subsequential *Uhlenbeck limits* which are Yang-Mills connections on bundles with a possibly different topology than the original E. The bundles are isometric, and the connections converge, away from a singular set of codimension four[3]. In dimension four, the singular set is a finite collection of points. For the precise definition, see Prop. 2.15 below. Our result is that on a Kähler surface, the Uhlenbeck limits are independent of the subsequence and are determined solely by the isomorphism class of the initial holomorphic bundle $(E, \bar{\partial}_E)$. More precisely:

Theorem 1 (Main Theorem). *Let X be a compact Kähler surface, $E \to X$ a hermitian vector bundle, and D_0 an integrable unitary connection on E inducing a holomorphic structure $\bar{\partial}_E = D_0''$. Let D_∞ denote the Yang-Mills connection on $\mathrm{Gr}_{\omega}^{\mathrm{hns}}(E, \bar{\partial}_E)^{**}$ referred to above. Let D_t be the time t solution to the Yang-Mills flow with initial condition D_0. Then as $t \to \infty$, D_t converges in the sense of Uhlenbeck to D_∞.*

We now give a sketch of the ideas involved in the proof of the Main Theorem and explain the organization of the paper. In Section 2 we lay out the definitions of the Harder-

[3] In this paper, convergence of connections will always be modulo real gauge equivalence.

Narasimhan-Seshadri filtration and its associated graded object. We review the Yang-Mills flow and the notion of an Uhlenbeck limit. We also discuss other Yang-Mills type functionals associated to invariant convex functions on the Lie algebra of the unitary group. These are closely related to L^p norms, and they will play an important role not only in distinguishing the various strata, but also because one actually cannot expect good L^2 behavior in the constructions that follow.

Much of the difficulty in proving the Main Theorem arises from the fact that the Harder-Narasimhan filtration is not necessarily given by subbundles. Since we have restricted our attention in this paper to surfaces, the individual factors in the filtration are themselves locally free, but the successive quotients may have point singularities. These points are essentially the locus where one can expect bubbling to occur along the flow—when the filtration is by subbundles one can show there is no bubbling—and they are therefore a fundamental aspect of the problem and not a mere technical annoyance.

In Section 3 we therefore analyze the degree to which the singularities in the filtration can be resolved by blowing up. To be more precise, we are interested in comparing the Harder-Narasimhan filtrations of $E \to X$ and $\pi^*(E) \to \hat{X}$, where $\pi : \hat{X} \to X$ is a sequence of monoidal transformations, and the Kähler metric ω_ε on $\hat{X}$ is an ε-perturbation by the components of the exceptional divisor of the pullback of the Kähler metric ω on X. In Thm. 3.1 we prove that under the assumption that the successive quotients of the Harder-Narasimhan filtration are stable, there is a resolution $\hat{X}$ such that for sufficiently small ε, the Harder-Narasimhan filtration of $\pi^*(E)$ is given by subbundles and its direct image by π coincides with the Harder-Narasimhan filtration of E. The situation is more complicated for semistable factors, and the resolution of the filtration by subbundles may not correspond to the Harder-Narasimhan filtration for any $\varepsilon > 0$.

Nevertheless, this analysis is sufficient for our purposes. In particular, we introduce the notion of an *L^p-approximate critical hermitian structure*. Roughly speaking, this is a smooth hermitian metric on a holomorphic bundle whose curvature in the direction of the Kähler form is close in the L^p sense to a critical value determined by its Harder-Narasimhan type (see Def. 3.9). We prove Thm. 3.11 which states that there exist L^p-approximate critical hermitian structures for all $1 \leqq p < \infty$. This result is an L^p version of a conjecture attributed to Kobayashi. Interestingly, the method does not seem to extend to $p = \infty$.

The first step in the proof of the Main Theorem is to determine the Harder-Narasimhan type of an Uhlenbeck limit. Since the Hermitian-Yang-Mills numbers are monotone along the flow one can show that for an initial condition which is a sufficiently close approximate critical hermitian structure, the Uhlenbeck limit of a sequence along the flow must have the correct Harder-Narasimhan type. Then a length decreasing argument for the Yang-Mills flow, which closely resembles Hartman's result for the harmonic map flow, implies that *any* initial condition must have Uhlenbeck limits of the correct type (see Thm. 4.1).

The second step in the proof of the Main Theorem is to show that the holomorphic structure on the Uhlenbeck limit coincides with the double dual of the associated graded sheaf of the Harder-Narasimhan-Seshadri filtration. The approach here is necessarily completely different from that of [D]. The main idea is to generalize an argument of Donaldson who constructs limiting holomorphic maps from the sequence of complex gauge transformations defined by the sequence of connections along the flow. Instead of a map on the entire initial bundle, we show that maps can be formed for each of the pieces in the filtration separately. The result then follows by an inductive argument.

The proof that the limiting holomorphic structure is the correct one is largely independent of the details of the flow. Indeed, we only use the fact that the Yang-Mills numbers of a descending sequence in a complex gauge orbit are absolutely minimizing. Since this result is in some way disjoint from Thm. 1, we formulate it separately (see Thm. 5.1). Recall that sequences minimize Yang-Mills energy $\mathrm{YM}(D)$ if and only if they minimize Hermitian-Yang-Mills energy $\mathrm{HYM}(D)$:

Theorem 2 (Minimizing sequences). *Let X be a compact Kähler surface, $E \to X$ a hermitian vector bundle, and D_0 an integrable unitary connection on E inducing a holomorphic structure $\bar\partial_E = D_0''$. Let $\vec\mu_0$ be the Harder-Narasimhan type of $(E, \bar\partial_E)$, and let D_∞ denote the Yang-Mills connection on $\mathrm{Gr}_\omega^{\mathrm{hns}}(E, \bar\partial_E)^{**}$. Suppose D_j is a sequence of smooth unitary connections in the complex gauge orbit of D_0 such that $\mathrm{HYM}(D_j) \to \mathrm{HYM}(\vec\mu_0)$ as $j \to \infty$. Then there is a subsequence (also denoted j) and a finite set of points $Z^{\mathrm{an.}} \subset X$ such that*:

(1) *E and $\mathrm{Gr}_\omega^{\mathrm{hns}}(E, \bar\partial_E)^{**}$ are $L^p_{2,\mathrm{loc.}}$-isometric on $X \backslash Z^{\mathrm{an.}}$ for all p.*

(2) *$D_j \to D_\infty$ in $L^2_{\mathrm{loc.}}$ away from $Z^{\mathrm{an.}}$.*

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2. Preliminaries

2.1. Stability and the Harder-Narasimhan filtration. Let X be a complex surface. The singular set $\mathrm{Sing}(E)$ of a coherent analytic torsion-free sheaf $E \to X$ is the closed subvariety where E fails to be locally free. Since $\dim_{\mathbb{C}} X = 2$, the singular set of a torsion-free sheaf is a locally finite collection of points and reflexive sheaves are locally free (cf. [Ko], Cor. V.5.15 and V.5.20). A subsheaf $S \subset E$ of a reflexive sheaf E is said to be saturated if the quotient $Q = E/S$ is torsion-free. In general, the saturation of a subsheaf S in E, denoted $\mathrm{Sat}_E(S)$, is the kernel of the sheaf map $E \to Q/\mathrm{Tor}(Q)$, where $\mathrm{Tor}(Q)$ is the torsion subsheaf of Q. Note that S is a subsheaf of $\mathrm{Sat}_E(S)$ with a torsion quotient. A saturated subsheaf of a reflexive sheaf is reflexive (cf. [Ko], Prop. V.5.22). We will also need the following result, whose proof is standard:

Lemma 2.1. *Let E be a torsion-free sheaf. Suppose $S_1 \subset S_2 \subset E$ are subsheaves with S_2/S_1 a torsion sheaf. Then $\mathrm{Sat}_E(S_1) = \mathrm{Sat}_E(S_2)$.*

Now assume that X is compact with a Kähler form ω. We will assume the volume of

X with respect to ω is normalized to be $\mathrm{vol}(X) = 2\pi$. The ω-slope $\mu(E)$ of a torsion-free sheaf $E \to X$ is defined by:

$$\mu_\omega(E) = \frac{\deg_\omega(E)}{\mathrm{rk}(E)} = \frac{1}{\mathrm{rk}(E)} \int_X c_1(E) \wedge \omega. \tag{2.1}$$

We define $\mu_{\max}(E)$ to be the maximal slope of a subsheaf of E, and $\mu_{\min}(E)$ to be the minimal slope of a torsion-free quotient of E. A torsion-free sheaf $E \to X$ is ω-stable (resp. ω-semistable) if for all subsheaves $F \subset E$ with $0 < \mathrm{rk}(F) < \mathrm{rk}(E)$, $\mu_\omega(F) < \mu_\omega(E)$ (resp. $\mu_\omega(F) \leqq \mu_\omega(E)$). When the Kähler form is understood we shall sometimes refer to E simply as stable or semistable, and we will also omit subscripts and write $\mu(E)$.

Proposition 2.2 (cf. [Ko], Thm. V.7.15). *Let $E \to X$ be a torsion-free sheaf. Then there is a filtration*: $0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E$, *called the Harder-Narasimhan filtration of E (abbr.* HN *filtration), such that $Q_i = E_i/E_{i-1}$ is torsion-free and semistable. Moreover, $\mu(Q_i) > \mu(Q_{i+1})$, and the associated graded object $\mathrm{Gr}^{\mathrm{hn}}_\omega(E) = \bigoplus_{i=1}^{\ell} Q_i$ is uniquely determined by the isomorphism class of E.*

It will be convenient to denote the subsheaf E_i in the HN filtration by $\mathbb{F}^{\mathrm{hn}}_i(E)$, or by $\mathbb{F}^{\mathrm{hn}}_{i,\omega}(E)$, when we wish to emphasize the role of the Kähler structure. The collection of slopes $\mu(Q_i)$ is an important invariant of the isomorphism class of a torsion-free sheaf. For a torsion-free sheaf E of rank R construct an R-tuple of numbers $\vec{\mu}(E) = (\mu_1, \ldots, \mu_R)$ from the HN filtration by setting: $\mu_i = \mu(Q_j)$, for $\mathrm{rk}(E_{j-1}) + 1 \leqq i \leqq \mathrm{rk}(E_j)$. We call $\vec{\mu}(E)$ the Harder-Narasimhan type of E. These invariants admit a natural partial ordering which will be very relevant to this paper. For a pair $\vec{\mu}, \vec{\lambda}$ of R-tuple's satisfying $\mu_1 \geqq \cdots \geqq \mu_R$, $\lambda_1 \geqq \cdots \geqq \lambda_R$, and $\sum_{i=1}^{R} \mu_i = \sum_{i=1}^{R} \lambda_i$, we define:

$$\vec{\mu} \leqq \vec{\lambda} \Leftrightarrow \sum_{j \leqq k} \mu_j \leqq \sum_{j \leqq k} \lambda_j, \quad \text{for all } k = 1, \ldots, R. \tag{2.2}$$

The importance of this ordering is that it defines a stratificaton of the space of holomorphic structures on a given complex vector bundle over a Riemann surface. See [AB], §7 for more details. We will make use of the following simple fact:

Lemma 2.3. *Let $\vec{\mu} = (\mu_1, \ldots, \mu_R)$ and $\vec{\lambda} = (\lambda_1, \ldots, \lambda_R)$ be nonincreasing R-tuples as above. Suppose there is a partition $0 = R_0 < R_1 < \cdots < R_\ell = R$ such that $\mu_i = \mu_j$ for all pairs i, j satisfying: $R_{k-1} + 1 \leqq i, j \leqq R_k$, $k = 1, \ldots, \ell$. If $\sum_{j \leqq R_k} \mu_j \leqq \sum_{j \leqq R_k} \lambda_j$, for all $k = 1, \ldots, \ell$, then $\vec{\mu} \leqq \vec{\lambda}$.*

Several technical properties of the HN filtration will also play a role in this paper. We again omit the proofs.

Proposition 2.4. (1) *Let $\tilde{E} \to X$ be torsion-free and $E \subset \tilde{E}$ with $T = \tilde{E}/E$ a torsion sheaf supported at points. Then $\mathbb{F}^{\mathrm{hn}}_i(E) = \ker\big(\mathbb{F}^{\mathrm{hn}}_i(\tilde{E}) \to T\big)$, and $\mathbb{F}^{\mathrm{hn}}_i(\tilde{E}) = \mathrm{Sat}_{\tilde{E}}\big(\mathbb{F}^{\mathrm{hn}}_i(E)\big)$.*

(2) *Let $E \to X$ be a torsion-free sheaf. Let $E_1 = \mathbb{F}^{\mathrm{hn}}_1(E)$, and $Q_1 = E/E_1$. Then*

$$\mathbb{F}^{\mathrm{hn}}_{i+1}(E) = \ker\big(E \to Q_1/\mathbb{F}^{\mathrm{hn}}_i(Q_1)\big).$$

In particular, $\mathbb{F}^{\mathrm{hn}}_{i+1}(E)/E_1 = \mathbb{F}^{\mathrm{hn}}_i(Q_1)$.

(3) *Consider an exact sequence*: $0 \to S \to E \to Q \to 0$ *of torsion-free sheaves on* X *with* $\mu_{\min}(S) > \mu_{\max}(Q)$. *Then the Harder-Narasimhan filtration of* E *is given by*:

$$0 = E_0 \subset \mathbb{F}_1^{\mathrm{hn}}(S) \subset \cdots \subset \mathbb{F}_k^{\mathrm{hn}}(S) = S \subset \mathbb{F}_{k+1}^{\mathrm{hn}}(E) \subset \cdots \subset \mathbb{F}_\ell^{\mathrm{hn}}(E) = E,$$

where $\mathbb{F}_i^{\mathrm{hn}}(E) = \mathbb{F}_i^{\mathrm{hn}}(S)$, $i \leqq k$, and $\mathbb{F}_{k+i}^{\mathrm{hn}}(E) = \ker\big(E \to Q/\mathbb{F}_i^{\mathrm{hn}}(Q)\big)$ *for* $i = 0, 1, \ldots, \ell - k$. *In particular*, $\mathrm{Gr}^{\mathrm{hn}}(E) \simeq \mathrm{Gr}^{\mathrm{hn}}(S) \oplus \mathrm{Gr}^{\mathrm{hn}}(Q)$.

We point out the analogous filtrations for semistable sheaves:

Proposition 2.5 (cf. [Ko], Thm. V.7.18). *Let* $Q \to X$ *be a semistable torsion-free sheaf. Then there is a filtration* $0 = F_0 \subset F_1 \subset \cdots \subset F_\ell = Q$, *called a Seshadri filtration of* E, *such that* F_i/F_{i-1} *is stable and torsion-free. Moreover*, $\mu(F_i/F_{i-1}) = \mu(Q)$ *for each i. The associated graded object* $\mathrm{Gr}_\omega^{\mathrm{s}}(Q) = \bigoplus_{i=1}^{\ell} F_i/F_{i-1}$, *is uniquely determined by the isomorphism class of* Q.

Finally, the double filtration whose associated graded sheaf appears in the statements of the main results in the Introduction is obtained by combining Prop.'s 2.2 and 2.5:

Proposition 2.6. *Let* $E \to X$ *be a torsion-free sheaf. Then there is a double filtration* $\{E_{i,j}\}$, *called a Harder-Narasimhan-Seshadri filtration of* E (*abbr.* HNS-*filtration*), *with the following properties*: *if* $\{E_i\}_{i=1}^{\ell}$ *is the* HN *filtration of* E, *then*

$$E_{i-1} = E_{i,0} \subset E_{i,1} \subset \cdots \subset E_{i,\ell_i} = E_i,$$

and the successive quotients $Q_{i,j} = E_{i,j}/E_{i,j-1}$ *are stable torsion-free sheaves. Moreover*, $\mu(Q_{i,j}) = \mu(Q_{i,j+1})$ *and* $\mu(Q_{i,j}) > \mu(Q_{i+1,j})$. *The associated graded object*:

$$\mathrm{Gr}_\omega^{\mathrm{hns}}(E) = \bigoplus_{i=1}^{\ell}\bigoplus_{j=1}^{\ell_i} Q_{i,j}$$

is uniquely determined by the isomorphism class of E.

2.2. Yang-Mills connections and Uhlenbeck limits. Given a smooth complex vector bundle $E \to X$ of rank R, let $\Omega^{p,q}(E)$ denote the space of smooth (p,q) forms with values in E. We will regard a holomorphic structure on E as given by a $\bar\partial$ operator $\bar\partial_E : \Omega^{p,q}(E) \to \Omega^{p,q+1}(E)$ satisfying the integrability condition $\bar\partial_E \circ \bar\partial_E = 0$. We will sometimes denote the holomorphic structure on E explicitly by $(E, \bar\partial_E)$. When this structure is understood, we will confuse the notation for the holomorphic bundle and the sheaf of holomorphic sections by E, as we have done in the previous section.

Now suppose we are given a smooth hermitian metric H on $(E, \bar\partial_E)$. Then there is a uniquely determined H-unitary connection D on E satisfying $D'' = \bar\partial_E$, where D'' denotes the $(0,1)$ part of D (D' will denote the $(1,0)$ part). We will sometimes denote this connection by $D = (\bar\partial_E, H)$. Conversely, given a unitary connection D on E whose curvature $F_D = D \circ D$ is of type $(1,1)$ (i.e. $F_D^{0,2} = 0$), then $D'' = \bar\partial_E$ defines a holomorphic structure on E, and $D = (D'', H)$.

For a fixed hermitian metric, let $\mathscr{A}_H$ denote the space of H-unitary connections D on E, and let $\mathscr{A}_H^{1,1}$ denote those satisfying $F_D^{0,2} = 0$. The discussion above gives an identification of $\mathscr{A}_H^{1,1}$ with the space $\mathscr{A}''$ of integrable $\bar{\partial}$-operators, or holomorphic structures, on E. We denote by $\mathfrak{G}$ the space of unitary gauge transformations acting on $\mathscr{A}_H$ by pulling back. Via the identification $\mathscr{A}_H^{1,1} \simeq \mathscr{A}''$, it is clear that we have an action on $\mathscr{A}_H^{1,1}$ by $\mathfrak{G}^{\mathbb{C}}$, the complexification of $\mathfrak{G}$. We call $\mathfrak{G}^{\mathbb{C}}$ the complex gauge group. Notice that $\mathfrak{G}^{\mathbb{C}}$ also acts on the space of hermitian metrics on E, where $g(H)$ is defined by: $g(H)(s_1, s_2) = H(gs_1, gs_2)$.

Since many norms will be used in this paper, let us emphasize the following: if $\mathbf{a}$ is a hermitian or skew-hermitian endomorphism on an R-dimensional hermitian vector space with eigenvalues $\{\lambda_1, \ldots, \lambda_R\}$, we set:

$$|\mathbf{a}| = \left\{\sum_{i=1}^{R} |\lambda_i|^2\right\}^{1/2}. \tag{2.3}$$

For a hermitian vector bundle E, let $\mathfrak{u}(E)$ denote the subbundle of $\operatorname{End} E$ consisting of skew-hermitian endomorphisms. If $\mathbf{a}$ is a section of $\mathfrak{u}(E)$, then $|\mathbf{a}|$ will denote the pointwise norm defined by (2.3).

Given a Kähler metric ω on X, the Yang-Mills functional (abbr. YM *functional*) is defined by $\mathrm{YM}(D) = \|F_D\|^2_{L^2(\omega)}$, and the Hermitian-Yang-Mills functional (abbr. HYM *functional*) is defined by $\mathrm{HYM}(D) = \|\Lambda_\omega F_D\|^2_{L^2(\omega)}$. Here, Λ_ω denotes contraction with the Kähler form, and $\Lambda_\omega F_D$ is called the Hermitian-Einstein tensor associated to D. Since for any $D \in \mathscr{A}_H^{1,1}$ we have (cf. [Ko], IV.3.29):

$$\mathrm{YM}(D) = \mathrm{HYM}(D) + 4\pi^2\big(2c_2(E) - c_1^2(E)\big), \tag{2.4}$$

the YM and HYM functionals have the same critical points on $\mathscr{A}_H^{1,1}$; namely, the Yang-Mills Connections $D^* F_D = 0$. We also note the Kähler identities:

$$D^* F_D = \sqrt{-1}(D' - D'')\Lambda_\omega F_D. \tag{2.5}$$

Given a holomorphic bundle $E \to (X, \omega)$, a hermitian metric H is called a Hermitian-Einstein metric if there is a constant μ such that $\sqrt{-1}\Lambda_\omega F_{(\bar{\partial}_E, H)} = \mu \mathbf{I}_E$, where $\mathbf{I}_E$ denotes the identity endomorphism of E. If X is compact, then because of the normalization $\operatorname{vol}(X) = 2\pi$ it is necessarily the case that $\mu = \mu_\omega(E)$ (see (2.1)). The celebrated theorem of Donaldson-Uhlenbeck-Yau relates stability to the existence of a Hermitian-Einstein metric (cf. [Do1], [Do2], [UY]):

Theorem 2.7. *A holomorphic vector bundle E on a compact Kähler manifold (X, ω) admits a Hermitian-Einstein metric if and only if it is holomorphically split into a direct sum of ω-stable bundles, all with slope $= \mu(E)$.*

The following is standard (cf. [AB], §4, or [Ko], IV. §3):

Proposition 2.8. *Let $D \in \mathscr{A}_H^{1,1}$ be a* YM *connection on a hermitian vector bundle over a Kähler manifold X. Then there is an orthogonal splitting $(E, D) = \bigoplus_{i=1}^{\ell} (Q_i, D_i)$, where*

$\sqrt{-1}\Lambda_\omega F_{D_i} = \mu_i \mathbf{I}_{Q_i}$, *for constants* μ_i. *In case* X *is compact*, $\mu_i = \mu(Q_i)$, *and the critical values of the* YM *functional on* $\mathscr{A}_H^{1,1}$ *are discrete.*

For the proof of the following version of Uhlenbeck compactness, see [U1] (and also, [UY], Thm. 5.2):

Proposition 2.9. *Let* X *be a compact Kähler surface and* $E \to X$ *a complex vector bundle with hermitian metric* H. *Assume* D_j *is a sequence of integrable unitary connections on* E *such that* $\|\Lambda F_{D_j}\|_{L^\infty}$ *is bounded uniformly for all* j. *Fix* $p > 4$. *Then there is*:

(1) *a subsequence* $\{j_k\}$,

(2) *a finite subset* $Z^{\mathrm{an.}} \subset X$,

(3) *a smooth hermitian vector bundle* $(E_\infty, H_\infty) \to X \backslash Z^{\mathrm{an.}}$ *with a finite action connection* D_∞ *on* E_∞,

(4) *for any compact set* $W \subset\subset X \backslash Z^{\mathrm{an.}}$, *an* L_2^p*-isometry*

$$\tau^W : (E_\infty, H_\infty)|_W \to (E, H)|_W.$$

such that for $W \subset W' \subset\subset X \backslash Z^{\mathrm{an.}}$, $\tau^W = \tau^{W'}|_W$, *and* $\tau^W(D_{j_k}) \rightharpoonup D_\infty$ *weakly*[4] *in* $L_1^p(W)$.

We will call any D_∞ arising in this way an *Uhlenbeck limit* of the sequence D_j. We will often omit the isometries τ^W from the notation, and simply identify (E_∞, H_∞) with (E, H) on $X \backslash Z^{\mathrm{an.}}$. Also, it is useful to note here that weak $L_{1,\mathrm{loc.}}^p$ convergence implies convergence of local holomorphic frames. This may be proven, for example, using Webster's proof of the Newlander-Nirenberg Theorem [W]. We refer to [DW] for more details on the following result:

Proposition 2.10. *Let* $D_j \rightharpoonup D_\infty$ *in* $L_{1,\mathrm{loc.}}^p(X \backslash Z^{\mathrm{an.}})$ *for some* $p > 4$, *as in Prop.* 2.9. *Then for each* $x \in X \backslash Z^{\mathrm{an.}}$ *there is*:

(1) *a coordinate neighborhood* $U \subset X \backslash Z^{\mathrm{an.}}$ *of* x,

(2) *a sequence* $\{\mathbf{s}_j\}$ *of* D_j''*-holomorphic frames on* U,

(3) *a* D_∞''*-holomorphic frame* $\mathbf{s}_\infty$ *on* U,

(4) *and a subsequence* $\{j_k\} \subset \{j\}$,

such that $\mathbf{s}_{j_k} \to \mathbf{s}_\infty$ *in* $C^1(U)$.

Next, we turn to a situation where the Uhlenbeck limits are Yang-Mills:

Proposition 2.11. *If in addition to the assumptions in Prop.* 2.9 *we assume* $\|D_j \Lambda F_{D_j}\|_{L^2} \to 0$, *then any Uhlenbeck limit* D_∞ *is Yang-Mills. Moreover, the triple*

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[4] We will denote weak convergence by "$\rightharpoonup$" and strong convergence by "$\to$".

$(E_\infty, D_\infty, H_\infty)$ extends smoothly to X, and the extension has a holomorphic orthogonal splitting as a direct sum:

$$\bigoplus_{i=1}^{\ell}(Q_\infty^{(i)}, D_\infty^{(i)}, H_\infty^{(i)}),$$

where $H_\infty^{(i)}$ is a Hermitian-Einstein metric on $Q_\infty^{(i)}$.

Proof. The last statement follows by the removable singularities theorem (cf. [U2]) and the argument cited above (cf. Prop. 2.8). To see that D_∞ is Yang-Mills, we argue as follows: by the compactness of the embedding $L_1^p \hookrightarrow C^0$ and the fact that $D_{j_k} \rightharpoonup D_\infty$ weakly in $L^p_{1,\mathrm{loc.}}$, we may assume $D_{j_k} \xrightarrow{C^0_{\mathrm{loc.}}} D_\infty$, and $\Lambda F_{D_{j_k}} \rightharpoonup \Lambda F_{D_\infty}$ in $L^p_{\mathrm{loc.}}$. It follows that $D_\infty \Lambda F_{D_{j_k}} \rightharpoonup D_\infty \Lambda F_{D_\infty}$ in $L^2_{-1,\mathrm{loc.}}$, say, where L^2_{-k} is the dual space to L^2_k. On the other hand, $D_\infty \Lambda F_{D_{j_k}} = D_{j_k} \Lambda F_{D_{j_k}} + [D_\infty - D_{j_k}, \Lambda F_{D_{j_k}}]$, so by the added hypothesis we also have, $D_\infty \Lambda F_{D_{j_k}} \xrightarrow{L^2_{\mathrm{loc.}}} 0$. This implies $D_\infty \Lambda F_{D_\infty} = 0$. The statement that D_∞ is Yang-Mills now follows from the Kähler identities (2.5). This completes the proof. □

Corollary 2.12. *With the assumptions as in Prop. 2.11, $\Lambda F_{D_{j_k}} \xrightarrow{L^p} \Lambda F_{D_\infty}$ for all $1 \leqq p < \infty$.*

Proof. Set $f_k = \Lambda F_{D_{j_k}} - \Lambda F_{D_\infty}$. Then by the proof of Prop. 2.11, $f_k \rightharpoonup 0$ in $L^p_{\mathrm{loc.}}$ and $D_\infty f_k \to 0$ strongly in $L^2_{\mathrm{loc.}}$. By Kato's inequality, $|f_k|$ is uniformly bounded in $L^2_{1,\mathrm{loc.}}$, so $|f_k| \xrightarrow{L^2_{\mathrm{loc.}}} 0$. Since $|f_k|$ is also uniformly bounded in L^∞, it follows that $|f_k| \xrightarrow{L^p} 0$ for all p. □

We conclude this subsection with a technical result on the boundedness of second fundamental forms. This will be important in the proof of the main result. Let $\Omega \subset X$ be an open set. Let $\{D_j\}, D_\infty$ be integrable unitary connections on $E \to \Omega$ with $D_j \rightharpoonup D_\infty$ in $L^p_{1,\mathrm{loc.}}(\Omega)$, for all p, $1 \leqq p < \infty$. Let π_j (resp. π_∞) be (smooth) projections onto D_j'' (resp. D_∞'') holomorphic subbundles of E. Let us assume the following:

(1) ΛF_{D_j} is bounded in $L^\infty_{\mathrm{loc.}}(\Omega)$ uniformly in j.

(2) π_j is bounded in $L^2_{1,\mathrm{loc.}}(\Omega)$ uniformly in j.

(3) $\|\pi_j - \pi_\infty\|_{L^\infty_{\mathrm{loc.}}(\Omega)} \to 0$ as $j \to \infty$.

Lemma 2.13. *With the assumptions above, π_j is bounded in $L^p_{2,\mathrm{loc.}}(\Omega)$ uniformly in j, for all p. In particular, the second fundamental forms $D_j''\pi_j$ are locally uniformly bounded.*

Proof. Set $p_j = \pi_j - \pi_\infty$. By [D], Lemma 3.2, we may write:

$$\Delta_{D_\infty}(p_j) = \{D_\infty p_j, D_\infty p_j\} + \{D_\infty \Gamma_j, p_j\} + \{D_\infty p_j, \Gamma_j\} + \{\Gamma_j, \Gamma_j\} + G_j, \tag{2.6}$$

where the brackets $\{\,,\}$ indicate a bilinear combination of the two arguments with bounded coefficients, $\Gamma_j = D_j - D_\infty$, and G_j is uniformly bounded in $L^\infty_{\mathrm{loc.}}(\Omega)$. Since Γ_j is uniformly bounded in $L^p_{1,\mathrm{loc.}}(\Omega)$ and $p_j \to 0$ in $L^\infty_{\mathrm{loc.}}(\Omega)$, it follows from [GiM], Thm. 1.4, and [Gi], Thm. VI.1.5, that p_j is bounded in $C^{1,\alpha}_{\mathrm{loc.}}(\Omega)$ for any $0 < \alpha < 1$, uniformly in j. The

$L^p_{2,\mathrm{loc.}}(\Omega)$ bound then follows from the L^p-elliptic estimate (cf. [GT], Thm. 9.11) applied to (2.6). For the second statement, write: $D_j''\pi_j = D_\infty''\pi_j + \Gamma_j''\pi_j$, and note that the right hand side is locally bounded since it is in $L^p_{1,\mathrm{loc.}}$ for $p > 4$. $\square$

2.3. The Yang-Mills flow. The basic object of interest in this paper is the Yang-Mills flow for a family of unitary connections $D = D(t) = D_t$. This is the L^2-gradient flow of the YM functional, which may be written as follows:

$$\frac{\partial D}{\partial t} = -D^* F_D, \quad D(0) = D_0 \in \mathscr{A}_H^{1,1}. \tag{2.7}$$

By the work of Donaldson and Simpson (cf. [Do1], [Si]), (2.7) has a unique solution in $(\mathscr{A}_H^{1,1}/\mathfrak{G}) \times [0,\infty)$. Furthermore, D_t lies in a single $\mathfrak{G}^{\mathbb{C}}$ orbit, namely $\mathfrak{G}^{\mathbb{C}} \cdot D_0$, for all $t \in [0,\infty)$. One way to see this is to fix the $\bar\partial$-operator $\bar\partial_E = D_0''$ on E and look at the family of hermitian metrics $H = H(t) = H_t$ satisfying the Hermitian-Yang-Mills flow equations:

$$H_t^{-1}\frac{\partial H_t}{\partial t} = -2(\sqrt{-1}\Lambda_\omega F_{H_t} - \mu \mathbf{I}_E), \quad H(0) = H_0. \tag{2.8}$$

In the equation above, F_{H_t} denotes the curvature of $D_t = (\bar\partial_E, H_t)$, and $\mu = \mu_\omega(E)$ (see (2.1)) depends only on the topology of E and the Kähler form ω. The two systems (2.7) and (2.8) are equivalent up to gauge. See [Do1] for more details. Also notice that it is easy to factor out the trace part of the connection and gauge transformations. Therefore, in the following we shall assume that *the solutions to the above equations all preserve determinants*. Also, in the following and throughout the paper, we will often omit the Kähler form ω from the notation when doing so is unambiguous.

The following is an immediate consequence of (2.7) and (2.8) (see [Do1], Prop. 16, for a proof):

Lemma 2.14. (1) *Let D_t be a solution to* (2.7). *Then $\partial F_{D_t}/\partial t = -\Delta_t F_{D_t}$, and*:

$$\frac{d}{dt}\|F_{D_t}\|^2_{L^2} = -2\|D_t^* F_{D_t}\|^2_{L^2} \leqq 0.$$

Hence, $t \mapsto \mathrm{YM}(D_t)$ *and* $t \mapsto \mathrm{HYM}(D_t)$ *are nonincreasing.*

(2) *The pointwise norm $|\Lambda F_{D_t}|^2$ satisifies* $\dfrac{\partial}{\partial t}|\Lambda F_{D_t}|^2 + \Delta|\Lambda F_{D_t}|^2 \leqq 0$.

Uhlenbeck compactness applied to the flow gives the following:

Proposition 2.15. *Let D_t be a solution to* (2.7) *on a compact Kähler surface X, and fix $p > 4$. For any sequence $t_j \to \infty$ we can find the following*:

(1) *a subsequence $\{t_{j_k}\}$*,

(2) *a finite set of points $Z^{\mathrm{an.}} \subset X$*,

(3) *a smooth hermitian vector bundle* $(E_\infty, H_\infty) \to X$ *with a finite action Yang-Mills connection* D_∞ *on* E_∞,

(4) *and on any compact set* $W \subset\subset X \backslash Z^{\text{an.}}$, *an* L_2^p*-isometry*

$$\tau^W : (E_\infty, H_\infty)|_W \to (E_0, H_0)|_W,$$

such that for $W \subset W' \subset\subset X \backslash Z^{\text{an.}}$, $\tau^W = g^{W'}|_W$, *and* $\tau^W(D_{t_{j_k}}) \rightharpoonup D_\infty$, *weakly in* $L_1^p(W)$. *Moreover, the triple* $(E_\infty, D_\infty, H_\infty)$ *extends smoothly to* X, *and the extension has a holomorphic orthogonal splitting as a direct sum*: $\bigoplus_{i=1}^{\ell}(Q_\infty^{(i)}, D_\infty^{(i)}, H_\infty^{(i)})$, *where* $H_\infty^{(i)}$ *is a Hermitian-Einstein metric on* $Q_\infty^{(i)}$.

Proof. By Lemma 2.14 (2) and the maximum principle, $\|\Lambda F_{D_t}\|_{L^\infty}$ is decreasing in t, and is therefore uniformly bounded. By [DoKr], Prop. 6.2.14, $\lim_{t\to\infty} \|D_t \Lambda F_{D_t}\|_{L^2} = 0$. The weak $L^p_{1,\text{loc.}}$ convergence along a subsequence to a YM connection now follows from Prop. 2.11. $\square$

As in §2.2, we will call D_∞ an Uhlenbeck limit of the flow. Note that a priori, D_∞ may depend on the choice of subsequence $\{t_{j_k}\}$, however we will see shortly that this is not the case.

Lemma 2.16. *Let* D_{t_j} *be a sequence of connections along the* YM *flow with Uhlenbeck limit* D_∞. *Then for* $t_j \geqq t_0 \geqq 0$, $\|\Lambda F_{D_\infty}\|_{L^\infty} \leqq \|\Lambda F_{D_{t_j}}\|_{L^\infty} \leqq \|\Lambda F_{D_{t_0}}\|_{L^\infty}$.

Proof. As stated above, $\|\Lambda F_{D_t}\|_{L^\infty}$ is decreasing in t. Fix $t \geqq 0$. Then for any $1 \leqq p < \infty$ and j sufficiently large we have:

$$\|\Lambda F_{D_{t_j}}\|_{L^p} \leqq (2\pi)^{1/p} \|\Lambda F_{D_{t_j}}\|_{L^\infty} \leqq (2\pi)^{1/p} \|\Lambda F_{D_t}\|_{L^\infty}$$

(recall $\operatorname{vol}(X) = 2\pi$). On the other hand, by Cor. 2.12, $\lim_{j\to\infty} \|\Lambda F_{D_{t_j}}\|_{L^p} = \|\Lambda F_{D_\infty}\|_{L^p}$, for all p. Hence, $\|\Lambda F_{D_\infty}\|_{L^p} \leqq (2\pi)^{1/p} \|\Lambda F_{D_t}\|_{L^\infty}$. Letting $p \to \infty$, we conclude

$$\|\Lambda F_{D_\infty}\|_{L^\infty} \leqq \|\Lambda F_{D_t}\|_{L^\infty}. \quad \square$$

Lemma 2.17. *If* D_∞ *is the Uhlenbeck limit of* D_{t_j}, *then* $\Lambda F_{D_{t_j}} \xrightarrow{L^p} \Lambda F_{D_\infty}$ *for all* $1 \leqq p < \infty$. *Moreover,* $\lim_{t\to\infty} \mathrm{HYM}(D_t) = \mathrm{HYM}(D_\infty)$.

Proof. The first part of the lemma follows from Cor. 2.12. The second part is immediate, since by Lemma 2.14, $t \mapsto \mathrm{HYM}(D_t)$ is nonincreasing, and

$$\mathrm{HYM}(D_{t_j}) \to \mathrm{HYM}(D_\infty). \quad \square$$

The Uhlenbeck limits obtained from the Yang-Mills flow are unique:

Proposition 2.18. *Let* D_t *be the solution to the* YM *flow* (2.7), *and suppose* D_∞ *is an Uhlenbeck limit for some sequence* D_{t_j} *with singular set* $Z^{\text{an.}}$. *Then* $D_t \to D_\infty$ *in* $L^2_{\text{loc.}}$ *away from* $Z^{\text{an.}}$. *In particular, the Uhlenbeck limit of the flow is uniquely defined up to gauge.*

Proof. By (2.4) and Lemma 2.17,

$$\lim_{t\to\infty} \mathrm{YM}(D_t) = \mathrm{HYM}(D_\infty) + 4\pi^2\big(2c_2(E) - c_1^2(E)\big).$$

From (2.7) and Lemma 2.14 (1), if $t_j \geqq t$:

$$\|D_{t_j} - D_t\|_{L^2}^2 \leqq \int_t^{t_j} \left\|\frac{\partial D_s}{\partial s}\right\|_{L^2}^2 ds = \int_t^{t_j} \|D_s^* F_{D_s}\|_{L^2}^2\, ds$$

$$= -\frac{1}{2}\int_t^{t_j} \frac{d}{ds}\|F_{D_s}\|_{L^2}^2\, ds = \frac{1}{2}\big(\mathrm{YM}(D_t) - \mathrm{YM}(D_{t_j})\big).$$

Since the limit of $\mathrm{YM}(D_t)$ as $t \to \infty$ exists, $\|D_{t_j} - D_t\|_{L^2} \to 0$ as t and $j \to \infty$. Since $D_{t_j} \xrightarrow{L^2_{\mathrm{loc.}}} D_\infty$, the convergence follows. For the last statement, if D_∞ and $\tilde{D}_\infty$ are two Uhlenbeck limits with singular sets $Z^{\mathrm{an.}}$ and $\tilde{Z}^{\mathrm{an.}}$, then the argument above shows that D_∞ and $\tilde{D}_\infty$ are gauge equivalent on $X \backslash Z^{\mathrm{an.}} \cup \tilde{Z}^{\mathrm{an.}}$. In particular, the holomorphic bundles (E_∞, D''_∞) and $(\tilde{E}_\infty, \tilde{D}''_\infty)$ are isomorphic on $X \backslash Z^{\mathrm{an.}} \cup \tilde{Z}^{\mathrm{an.}}$. But then their reflexive extensions are isomorphic as well by Hartogs theorem. □

Remark 2.19. In light of Prop. 2.18, we may speak of *the* Uhlenbeck limit of the flow D_t. Note, however, that we have not established that the singular set $Z^{\mathrm{an.}}$ is independent of the subsequence $\{t_j\}$.

We next turn to the HN type of the Uhlenbeck limit:

Lemma 2.20. *Let $D_j = g_j(D_0)$ be a sequence of complex gauge equivalent integrable connections on a complex vector bundle E of rank R with hermitian metric H_0. Let S be a coherent subsheaf of (E, D''_0) of rank r. Suppose that $\sqrt{-1}\Lambda F_{D_j} \xrightarrow{L^1} \mathbf{a}$, where $\mathbf{a} \in L^1\big(\sqrt{-1}\mathfrak{u}(E)\big)$, and that the eigenvalues $\lambda_1 \geqq \cdots \geqq \lambda_R$ of $\mathbf{a}$ (counted with multiplicities) are constant almost everywhere. Then:* $\deg(S) \leqq \sum_{i\leqq r} \lambda_i$.

Proof. Since $\deg(S) \leqq \deg\big(\mathrm{Sat}_E(S)\big)$, we may assume that S is saturated. Let π_j denote the orthogonal projection onto $g_j(S)$ with respect to the hermitian metric H_0. This is a bounded measurable hermitian endomorphism of E, smooth away from the singular set of E/S. The condition of being a weakly holomorphic projection implies $\pi_j^2 = \pi_j = \pi_j^*$, $\pi_j^\perp D''_j \pi_j = 0$, where $\pi_j^\perp = \mathbf{I}_E - \pi_j$. In fact, the π_j are L_1^2 sections of the smooth endomorphism bundle of E (cf. [UY], §4), and conversely, any such π defines a unique saturated subsheaf[5]. Moreover, the usual degree formula applies (see [Si], Lemma 3.2), so:

$$\deg(S) = \frac{1}{2\pi}\int_X \big(\mathrm{Tr}(\sqrt{-1}\Lambda F_{D_j}\pi_j) - |D''_j\pi_j|^2\big)\, dvol \leqq \frac{1}{2\pi}\int_X \mathrm{Tr}(\sqrt{-1}\Lambda F_{D_j}\pi_j)\, dvol \tag{2.9}$$

$$= \frac{1}{2\pi}\int_X \mathrm{Tr}(\mathbf{a}\pi_j)\, dvol + \frac{1}{2\pi}\int_X \mathrm{Tr}\big((\sqrt{-1}\Lambda F_{D_j} - \mathbf{a})\pi_j\big)\, dvol.$$

We now use the following result from linear algebra:

[5] We will often confuse the notation π of the projection operator with the subsheaf it defines.

Claim. *Let V be a finite dimensional hermitian vector space of complex dimension R, $L \in \mathrm{End}(V)$ a hermitian operator with eigenvalues $\lambda_1 \geqq \cdots \geqq \lambda_R$ (counted with multiplicities). Let $\pi = \pi^2 = \pi^*$ denote the orthogonal projection onto a subspace of dimension r. Then $\mathrm{Tr}(L\pi) \leqq \sum_{i \leqq r} \lambda_i$.*

Proof (sketch). Let $\{e_i\}_{i=1}^R$ be a unitary basis with $Le_i = \lambda e_i$. If we set $\alpha_i = \|\pi e_i\|^2$, then the claim follows by showing that the affine function $F(\alpha_1, \ldots, \alpha_R) = \sum_{i \leqq r} \lambda_i - \sum_{i=1}^R \lambda_i \alpha_i$, is nonnegative on the affine set $0 \leqq \alpha_i \leqq 1$, $\sum_{i=1}^R \alpha_i = r$. This may be proven by considering the extreme values of the α_i's and using induction on r and R. We omit the details. □

Given the claim, along with the fact that $\|\pi_j\|_{L^\infty} \leqq 1$, and the normalization $\mathrm{vol}(X) = 2\pi$, it follows from (2.9) that $\deg(S) \leqq \sum_{i \leqq r} \lambda_i + (1/2\pi)\|\sqrt{-1}\Lambda F_{D_j} - \mathbf{a}\|_{L^1}$. Now let $j \to \infty$ in this inequality to complete the proof of the lemma. □

Recall the partial ordering (2.2) of HN types of holomorphic structures on E:

Proposition 2.21. *Let D_j be a sequence along the* YM *flow on a bundle E of rank R with Uhlenbeck limit D_∞. Let $\vec{\mu}_0 = (\mu_1, \ldots, \mu_R)$ be the* HN *type of E with the holomorphic structure D_0'', and let $\vec{\lambda}_\infty = (\lambda_1, \ldots, \lambda_R)$ the type of D_∞''. Then $\vec{\mu}_0 \leqq \vec{\lambda}_\infty$.*

Proof. Let $\{E_i\}_{i=1}^\ell$ be the HN filtration of D_0''. Then $\deg(E_i) = \sum_{j \leqq \mathrm{rk}(E_i)} \mu_j$. By Lemma 2.17, $\Lambda F_{D_j} \xrightarrow{L^1} \Lambda F_{D_\infty}$. The type $\vec{\lambda}_\infty$ corresponds to the (constant) eigenvalues of ΛF_{D_∞}. Lemma 2.20 applied to $S = E_i$ implies $\deg(E_i) \leqq \sum_{j \leqq \mathrm{rk}(E_i)} \lambda_j$, for each $i = 1, \ldots, \ell$. The proposition now follows from Lemma 2.3. □

The following generalizes a result in [AB] to Kähler surfaces:

Corollary 2.22. *Let $\vec{\mu} = (\mu_1, \ldots, \mu_R)$ be the Harder-Narasimhan type of a rank R holomorphic vector bundle $(E, \bar{\partial}_E)$ on X. Then for all unitary connections D in the $\mathfrak{G}^{\mathbb{C}}$ orbit of $(E, \bar{\partial}_E)$:*

$$\sum_{i=1}^R \mu_i^2 \leqq \frac{1}{2\pi} \int_X |\Lambda F_D|^2 \, dvol.$$

Proof. Let D_t denote the YM flow with initial condition D. By Lemma 2.14 (2):

$$\int_X |\Lambda F_{D_t}|^2 \, dvol \leqq \int_X |\Lambda F_D|^2 \, dvol, \tag{2.10}$$

for all $t \geqq 0$. Let D_∞ be the Uhlenbeck limit along a subsequence $t_j \to \infty$. By Lemma 2.17:

$$\int_X |\Lambda F_{D_\infty}|^2 \, dvol = \lim_{j \to \infty} \int_X |\Lambda F_{D_{t_j}}|^2 \, dvol. \tag{2.11}$$

Now $\sqrt{-1}\Lambda F_{D_\infty}$ has constant eigenvalues $\vec{\lambda}_\infty$ which satisfy $\vec{\mu} \leqq \vec{\lambda}_\infty$ by Prop. 2.21. It follows from [AB], 12.6 (see also Prop. 2.24 below) that $\sum_{i=1}^{R}\mu_i^2 \leqq \sum_{i=1}^{R}\lambda_i^2$. This fact, along with (2.10) and (2.11), prove the result. □

2.4. Other Hermitian-Yang-Mills type functionals. One of the technical difficulties in dealing with holomorphic vector bundles of rank bigger than two is that the Hermitian-Yang-Mills numbers do not distinguish the different critical levels (or equivalently, different Harder-Narasimhan types) of the functionals YM and HYM. This was resolved by Atiyah and Bott in the case of vector bundles over Riemann surfaces by introducing Yang-Mills type functionals corresponding to higher symmetric functions of the eigenvalues of $\sqrt{-1} * F_D$ (cf. [AB], §8). In the case of vector bundles over higher dimensional Kähler manifolds there are analytic restrictions on the type of functionals we may consider. In this subsection we will explain in some detail how to use these functionals in order to distinguish the different critical levels.

Let $\mathfrak{u}(R)$ denote the Lie algebra of the unitary group $\mathrm{U}(R)$. Fix a real number $\alpha \geqq 1$. Then for $\mathbf{a} \in \mathfrak{u}(R)$, a skew hermitian matrix with eigenvalues $\sqrt{-1}\lambda_1, \dots, \sqrt{-1}\lambda_R$, let $\varphi_\alpha(\mathbf{a}) = \sum_{j=1}^{R} |\lambda_j|^\alpha$. It is easy to see that we can find a family $\varphi_{\alpha,\rho}$, $0 < \rho \leqq 1$, of smooth convex ad-invariant functions such that $\varphi_{\alpha,\rho} \to \varphi_\alpha$ uniformly on compact subsets of $\mathfrak{u}(R)$ as $\rho \to 0$. Hence, by [AB], Prop. 12.16, it follows that φ_α is a convex function on $\mathfrak{u}(R)$. For a given number N, define:

$$\mathrm{HYM}_{\alpha,N}(D) = \int_X \varphi_\alpha(\Lambda F_D + \sqrt{-1} N \mathbf{I}_E)\, dvol, \tag{2.12}$$

and $\mathrm{HYM}_\alpha(D) = \mathrm{HYM}_{\alpha,0}(D)$. Notice that $\mathrm{HYM} = \mathrm{HYM}_2$ is the ordinary HYM functional. Also, by a slight abuse of notation, we will set

$$\mathrm{HYM}_{\alpha,N}(\vec{\mu}) = \mathrm{HYM}_\alpha(\vec{\mu} + N) = 2\pi\varphi_\alpha\big(\sqrt{-1}(\vec{\mu} + N)\big),$$
$$\text{where} \quad \vec{\mu} + N = (\mu_1 + N, \dots, \mu_R + N)$$

is identified with the diagonal matrix $\mathrm{diag}(\mu_1 + N, \dots, \mu_R + N)$. In particular:

$$\mathrm{HYM}(\vec{\mu}) = 2\pi \sum_{i=1}^{R} \mu_i^2. \tag{2.13}$$

Lemma 2.23. *The functional* $\mathbf{a} \mapsto \Big(\int_X \varphi_\alpha(\mathbf{a})\, dvol\Big)^{1/\alpha}$, *defines a norm on* $L^\alpha\big(\mathfrak{u}(E)\big)$ *which is equivalent to the* L^α *norm.*

Proof. First, notice that there are universal constants C, C' (depending on R) such that for any real numbers $\lambda_1, \dots, \lambda_R$, and $\alpha \geqq 1$:

$$\frac{1}{C}\Big(\sum_{i=1}^{R}|\lambda_i|^2\Big)^{\alpha/2} \leqq \frac{1}{C}\Big(\sum_{i=1}^{R}|\lambda_i|\Big)^{\alpha} \leqq \sum_{i=1}^{R}|\lambda_i|^\alpha \leqq C\Big(\sum_{i=1}^{R}|\lambda_i|\Big)^{\alpha} \leqq C'\Big(\sum_{i=1}^{R}|\lambda_i|^2\Big)^{\alpha/2}.$$

Applying this to the eigenvalues of $\mathbf{a}$ and by integrating over X:

$$\frac{1}{C}\int_X (\operatorname{Tr}\mathbf{a}\mathbf{a}^*)^{\alpha/2}\,dvol \leqq \int_X \varphi_\alpha(\mathbf{a})\,dvol \leqq C'\int_X (\operatorname{Tr}\mathbf{a}\mathbf{a}^*)^{\alpha/2}\,dvol.$$

The lemma follows. □

We will require three important properties of the functionals $\mathrm{HYM}_{\alpha,N}$:

Proposition 2.24. (1) *If $\vec{\mu} \leqq \vec{\lambda}$, then $\varphi_\alpha(\sqrt{-1}\vec{\mu}) \leqq \varphi_\alpha(\sqrt{-1}\vec{\lambda})$ for all $\alpha \geqq 1$.*

(2) *Assume $\mu_R \geqq 0$ and $\lambda_R \geqq 0$. If $\varphi_\alpha(\sqrt{-1}\vec{\mu}) = \varphi_\alpha(\sqrt{-1}\vec{\lambda})$ for all α in some set $A \subset [1,\infty)$ possessing a limit point, then $\vec{\mu} = \vec{\lambda}$.*

Proof. (1) follows from [AB], 12.6. For (2), consider $f(\alpha) = \varphi_\alpha(\sqrt{-1}\vec{\lambda})$ and $g(\alpha) = \varphi_\alpha(\sqrt{-1}\vec{\mu})$ as functions of α. As complex valued functions, f, g clearly have analytic extensions to $\mathbb{C}\backslash\{\alpha \leqq 0\}$. Suppose that $f(\alpha) = g(\alpha)$ for all $\alpha \in A$. Then by analyticity, $f(\alpha) = g(\alpha)$ for all $\alpha \in \mathbb{C}\backslash\{\alpha \leqq 0\}$. If $\vec{\mu} \neq \vec{\lambda}$, then there is some k, $1 \leqq k \leqq R$, such that $\mu_i = \lambda_i$ for $i < k$, and $\mu_k \neq \lambda_k$; say, $\mu_k > \lambda_k$. Then for any $\alpha > 0$:

$$\left(\frac{\mu_k}{\lambda_k}\right)^\alpha \leqq \sum_{i=k}^R \left(\frac{\mu_i}{\lambda_k}\right)^\alpha = \sum_{i=k}^R \left(\frac{\lambda_i}{\lambda_k}\right)^\alpha \leqq R.$$

Letting $\alpha \to \infty$, we obtain a contradiction; hence, the result. □

Proposition 2.25. *Let D_t be a solution of* (2.7). *Then for any $\alpha \geqq 1$ and any N, $t \mapsto \mathrm{HYM}_{\alpha,N}(D_t)$ is nonincreasing.*

Proof. Because we can approximate φ_α by smooth convex ad-invariant functions $\varphi_{\alpha,\rho} \to \varphi_\alpha$, it suffices to show that the functional $t \mapsto \int_X \varphi_{\alpha,\rho}(\Lambda F_{D_t} + \sqrt{-1}N\mathbf{I}_E)\,dvol$, is nonincreasing along the flow for any $\rho > 0$. This follows from integrating the following inequality:

$$(2.14)\qquad (\partial/\partial t)\varphi_{\alpha,\rho}(\Lambda F_{D_t} + \sqrt{-1}N\mathbf{I}_E) + \Delta\varphi_{\alpha,\rho}(\Lambda F_{D_t} + \sqrt{-1}N\mathbf{I}_E) \leqq 0.$$

To prove (2.14), simplify the notation by setting $f = \Lambda F_{D_t} + \sqrt{-1}N\mathbf{I}_E$, $\varphi = \varphi_{\alpha,\rho}$. We first claim that:

$$(2.15)\qquad \Delta(\varphi \circ f)(x) = -*\varphi''_{f(x)}(*D_t f, D_t f) + \varphi'_{f(x)}(\Delta_{D_t} f).$$

Indeed, $\Delta(\varphi \circ f)(x) = -*d*d(\varphi \circ f)(x) = -*d*\varphi'_{f(x)}(D_t f)$, because φ is invariant under the adjoint action. Then:

$$\begin{aligned} -*d*\varphi'_{f(x)}(D_t f) &= -*d\big(\varphi'_{f(x)}(*D_t f)\big) = -*\varphi''_{f(x)}(*D_t f, D_t f) - *\varphi'_{f(x)}(D_t * D_t f) \\ &= -*\varphi''_{f(x)}(*D_t f, D_t f) + \varphi'_{f(x)}(\Delta_{D_t} f). \end{aligned}$$

The claim (2.15) follows. Since φ'' is a positive definite quadratic form:

$$\begin{aligned}\Delta\big(\varphi\circ(\Lambda F_{H_t}+\sqrt{-1}N\mathbf{I}_E)\big)(x) &\leqq \varphi'\big(\Delta_{D_t}(\Lambda F_{D_t}+\sqrt{-1}N\mathbf{I}_E)\big)\\ &= -\varphi'\left(\frac{\partial}{\partial t}(\Lambda F_{D_t}+\sqrt{-1}N\mathbf{I}_E)\right)\\ &= -\frac{\partial}{\partial t}\varphi(\Lambda F_{D_t}+\sqrt{-1}N\mathbf{I}_E),\end{aligned}$$

where the first equality follows from Lemma 2.14 (1). This verifies (2.14) and completes the proof. □

Proposition 2.26. *Let D_∞ be a subsequential Uhlenbeck limit of D_t, where D_t is a solution to* (2.7). *Then for any $\alpha \geqq 1$ and any N, $\lim_{t\to\infty}\mathrm{HYM}_{\alpha,N}(D_t) = \mathrm{HYM}_{\alpha,N}(D_\infty)$.*

Proof. Let D_∞ be the Uhlenbeck limit of a sequence D_{t_j}. By Lemma 2.17, $\Lambda F_{D_{t_j}} \xrightarrow{L^p} \Lambda F_{D_\infty}$ for all p. Hence, by Lemma 2.23, $\mathrm{HYM}_{\alpha,N}(D_{t_j}) \to \mathrm{HYM}_{\alpha,N}(D_\infty)$. The convergence in general follows by Prop. 2.25, since $\mathrm{HYM}_{\alpha,N}(D_t)$ is nonincreasing in t. □

3. Blow-up of the Harder-Narasimhan filtration and approximate metrics

The first goal of this section is to show how to resolve the Harder-Narasimhan filtration of a holomorphic bundle $E \to X$ by passing to a modification $\pi : \hat{X} \to X$ of the Kähler surface X. We shall see that this procedure works well when the associated graded object consists of stable sheaves. While this result is not directly needed for the remaining sections, we have chosen to present it here since it may be of independent interest. In the case where semistable quotients appear, the relationship between the HN filtration of E and that of π^*E is more complicated, and we have not attempted a complete description. As pointed out by the referee, much of this analysis has already appeared in the work of Buchdahl [Bu1], [Bu2].

The second goal is to show that there is a Hermitian metric $\hat{H}$ on π^*E so that the Hermitian-Yang-Mills numbers of $(\pi^*E, \hat{H})$ with respect to a natural family of Kähler metrics ω_ε are arbitrarily close (in an appropriate norm) to the slopes of $\mathrm{Gr}_\omega^{\mathrm{hn}}(E)$ on X for all ε sufficiently small. This is an important first step toward finding an approximate critical hermitian structure on X itself. We formulate this latter result below in Thm. 3.11. The argument we give here circumvents the need for an explicit description of the HN filtration of π^*E.

3.1. Resolution of the Harder-Narasimhan filtration. By Prop. 2.2, a holomorphic bundle E admits a filtration by saturated subsheaves E_i so that the successive quotients $Q_i = E_i/E_{i+1}$ are semistable and torsion-free. It follows that the E_i are locally free sheaves, i.e. vector bundles. They may, however, fail to be subbundles at finitely many points. Equivalently, the quotients Q_i are not necessarily locally free. For each i, we have an exact sequence of sheaves: $0 \to Q_i \to Q_i^{**} \to T_i \to 0$, where Q_i^{**} is locally free and T_i is a torsion sheaf supported at finitely many points. Define the set Z_i to be the support of T_i, and let $Z^{\mathrm{alg.}} = \bigcup_{i=1}^{k} Z_i$. We will refer to $Z^{\mathrm{alg.}}$ as the singular set of the filtration $\{E_i\}$ (in this paper we ignore multiplicities). We will prove the following:

Theorem 3.1. *Let $E \to X$ be a holomorphic vector bundle over a smooth Kähler surface X with Kähler form ω. Let $\mathrm{Gr}_\omega^{\mathrm{hn}}(E) = \bigoplus_{i=1}^{k} Q_i$ be the ω-HN filtration of E. We assume that the Q_i are stable. Then there is a smooth surface $\hat{X}$ obtained from X by a sequence of monoidal transformations $\pi : \hat{X} \to X$ with exceptional set $\mathbf{e} \subset \hat{X}$ satisfying the following properties*:

(1) $\pi(\mathbf{e}) = Z^{\mathrm{alg.}}$.

(2) *There exists a smooth, closed $(1,1)$ form η on $\hat{X}$ and a number $\varepsilon_0 > 0$ such that $\omega_\varepsilon = \pi^*\omega + \varepsilon\eta$ is a family of Kähler metrics on $\hat{X}$ for all $\varepsilon_0 \geqq \varepsilon > 0$.*

(3) *There is a number ε_1, $\varepsilon_0 \geqq \varepsilon_1 > 0$ such that for every $0 < \varepsilon \leqq \varepsilon_1$, the ω_ε-HN filtration $\{\mathbb{F}_{i,\omega_\varepsilon}^{\mathrm{hn}}(\hat{E})\}$ of $\hat{E} = \pi^*E$ is independent of ε and is a filtration by subbundles.*

(4) *For $0 < \varepsilon \leqq \varepsilon_1$, ε_1 as above, $\mathbb{F}_{i,\omega_\varepsilon}^{\mathrm{hn}}(\hat{E}) = \mathrm{Sat}_{\hat{E}}\big(\pi^*\mathbb{F}_i^{\mathrm{hn}}(E)\big)$. Moreover, if we write: $\mathrm{Gr}_{\omega_\varepsilon}^{\mathrm{hn}}(\hat{E}) = \bigoplus_{i=1}^{\hat{k}} \hat{Q}_i$, then for $0 < \varepsilon \leqq \varepsilon_1$, $\hat{k} = k$, and $(\pi_*\hat{Q}_i)^{**} \simeq Q_i^{**}$.*

Remark 3.2. The assumption that Q_i is stable is necessary. One can find examples where the pull-back of a semistable bundle is unstable for all $\varepsilon > 0$. More generally, the resolution of the Harder-Narasimhan filtration of E may not coincide with the Harder-Narasimhan filtration of π^*E for any $\varepsilon > 0$.

We begin by comparing stability of sheaves $\hat{E} \to \hat{X}$ with stability of their direct images $E = \pi_*\hat{E}$ on X. To do this, we need to define Kähler metrics. First, consider the case where $\hat{X}$ is the blow-up of X at a point and $\mathbf{e}$ is the exceptional divisor. Then there is a smooth, closed form η of type $(1,1)$ in the class of $c_1\big(\mathcal{O}_{\hat{X}}(-\mathbf{e})\big)$ on $\hat{X}$ such that $\omega_\varepsilon = \pi^*\omega + \varepsilon\eta$ is positive for all $\varepsilon > 0$ sufficiently small. This can be constructed quite explicitly (cf. [GH], pp. 182–187). In general, since $\hat{X}$ is a sequence of blow-ups at points, we can construct a family of Kähler forms on $\hat{X}$ by iterating the above argument. We state this precisely as:

Lemma 3.3. *Let $\pi : \hat{X} \to X$ be a sequence of monoidal transformations with exceptional set $\mathbf{e}$, and choose a Kähler form ω on X. Then there is a smooth, closed $(1,1)$ form η on $\hat{X}$ and a number $\varepsilon_0 > 0$ with the following properties*:

(1) *$\omega_\varepsilon = \pi^*\omega + \varepsilon\eta$ is a Kähler form on $\hat{X}$ for all $\varepsilon_0 \geqq \varepsilon > 0$.*

(2) *For any closed 2-form α on X, $\int_{\hat{X}} \pi^*\alpha \wedge \eta = 0$.*

Consider a family of Kähler forms in the manner of Lemma 3.3. Note that we do not normalize the volume of $(\hat{X}, \omega_\varepsilon)$, though we still assume the normalization on (X, ω); so $\mathrm{vol}(\hat{X}, \omega_\varepsilon) \to 2\pi$ as $\varepsilon \to 0$. In the following, let us agree that the slope $\mu(E)$ of a sheaf on X will be taken with respect to ω. For a sheaf $\hat{E}$ on $\hat{X}$, we denote by $\mu_\varepsilon(\hat{E})$ the slope of $\hat{E}$ with respect to the metric ω_ε. Similarly, a subscript ε will indicate that the quantity in question is taken with respect to ω_ε. With this understood, we have the following:

Proposition 3.4. *Given a holomorphic vector bundle $\hat{E} \to \hat{X}$ with $E = \pi_* \hat{E}$, and given $\delta > 0$, there is $\varepsilon_1 > 0$, depending upon $\hat{E}$, such that for all $0 < \varepsilon \leqq \varepsilon_1$ we have the following inequalities*:

(1) $\mu(E) - \delta \leqq \mu_\varepsilon(\hat{E}) \leqq \mu(E) + \delta$,

(2) $\mu_{\max}(E) - \delta \leqq \mu_{\max,\varepsilon}(\hat{E}) \leqq \mu_{\max}(E) + \delta$,

(3) $\mu_{\min}(E) - \delta \leqq \mu_{\min,\varepsilon}(\hat{E}) \leqq \mu_{\min}(E) + \delta$.

Proof. Since $\mu_{\min}(E) = -\mu_{\max}(E^*)$, part (3) will follow from part (2) applied to $\hat{E}^*$. Parts (1) and (2) are essentially contained in [Bu1], Lemma 5. The statement there assumed $\hat{E}$ is a pull-back bundle, but the proof works as well for general $\hat{E}$. $\square$

As a consequence, we have (cf. [Bu2], Prop. 3.4 (d)):

Corollary 3.5. *Let $\hat{E} \to \hat{X}$ and $E = \pi_* \hat{E}$ be as above. If E is ω-stable, then there is a number $\varepsilon_2 > 0$, depending upon $\hat{E}$, such that $\hat{E}$ is ω_ε-stable for all $0 < \varepsilon \leqq \varepsilon_2$.*

An inductive argument repeatedly using Prop. 3.4 implies convergence of the HN type:

Corollary 3.6. *Let $\hat{E} \to \hat{X}$ be a holomorphic vector bundle with $E = \pi_* \hat{E}$. Let $\vec{\mu}_\varepsilon$ denote the* HN *type of $\hat{E}$ with respect to ω_ε and $\vec{\mu}$ the* HN *type of E with respect to ω. Then $\vec{\mu}_\varepsilon \to \vec{\mu}$ as $\varepsilon \to 0$.*

Next, we state a general result on resolution of filtrations:

Proposition 3.7. *Let $0 = E_0 \subset E_1 \subset \cdots \subset E_{\ell-1} \subset E_\ell = E$ be a filtration of a holomorphic vector bundle $E \to X$ by saturated subsheaves E_i, and set $Q_i = E_i/E_{i-1}$. Then there is a sequence of monoidal transformations $\pi : \hat{X} \to X$ with exceptional set* $\mathbf{e}$ *and a filtration $0 = \hat{E}_0 \subset \hat{E}_1 \subset \cdots \subset \hat{E}_{\ell-1} \subset \hat{E}_\ell = \hat{E} = \pi^* E$, such that each $\hat{E}_i = \mathrm{Sat}_{\hat{E}}(\pi^* E_i)$ is a subbundle of $\hat{E}$. If we let $\hat{Q}_i = \hat{E}_i/\hat{E}_{i-1}$, we also have exact sequences $0 \to Q_i \to \pi_* \hat{Q}_i \to T_i \to 0$, where T_i is a torsion sheaf supported at the singular set of Q_i. Moreover, $\pi(\mathbf{e}) = Z^{\mathrm{alg.}}$, the union of the singular sets of Q_i; $\pi_* \hat{E}_i = E_i$; and $Q_i^{**} = (\pi_* \hat{Q}_i)^{**}$.*

Proof. The proof is standard resolution of singularities (cf. [Bu1], §3, for the step 2 filtration; the general argument then follows by induction). The form $\hat{E}_i = \mathrm{Sat}_{\hat{E}}(\pi^* E_i)$ follows from Lemma 2.1. The remaining statements follow easily, and we omit the details. $\square$

Proposition 3.8. *Let $\pi : \hat{X} \to X$ be a sequence of monoidal transformations with exceptional set* $\mathbf{e}$ *as above. Let $\hat{E} \to \hat{X}$ be a holomorphic vector bundle, and set $E = \pi_* \hat{E}$. Let $E_i = \mathbb{F}_i^{\mathrm{hn}}(E)$ denote the* HN *filtration of E, and assume that the successive quotients E_i/E_{i-1} are stable. Let $\hat{\jmath} : \pi^* E_i \to \hat{E}$ denote the induced map. We also assume that the sheaves $\mathrm{Sat}_{\hat{E}}(\hat{\jmath}(\pi^* E_i))$ are subbundles of $\hat{E}$. Then for $\varepsilon > 0$ sufficiently small, the* HN *filtration $\{\mathbb{F}_{i,\varepsilon}^{\mathrm{hn}}(\hat{E})\}$ with respect to the Kähler metrics of Lemma* 3.3 *is independent of ε and is given by $\mathbb{F}_{i,\varepsilon}^{\mathrm{hn}}(\hat{E}) = \mathrm{Sat}_{\hat{E}}(\hat{\jmath}(\pi^* E_i))$.*

Proof. We will proceed by induction on the length of the HN filtration of E. If E is stable, then the result follows from Cor. 3.5. Assume now that E is not stable, and *define* $\hat{E}_i = \mathrm{Sat}_{\hat{E}}(\jmath(\pi^* E_i))$. Note that with this definition it follows as in the proof of Prop. 3.4 that $\pi_* \hat{E}_i = E_i$. We claim that $\hat{E}_1 = \mathbb{F}^{\mathrm{hn}}_{1,\varepsilon}(\hat{E})$ for ε sufficiently small. This follows because (1) by the hypothesis and Cor. 3.5, $\hat{E}_1$ is stable for ε sufficiently small, and (2) we may arrange that $\mu_\varepsilon(\hat{E}_1) > \mu_{\max}(\hat{Q}_1)$ for ε sufficiently small, where $\hat{Q}_1 = \hat{E}/\hat{E}_1$. The claim then follows from Prop. 2.4 (3). Let $Q_1 = E/E_1$. By pushing forward, we have $0 \to Q_1 \to \tilde{Q}_1 \to T \to 0$, where $\tilde{Q}_1 = \pi_* \hat{Q}_1$, and T is a torsion-sheaf supported at points. Hence, by Prop. 2.4 (1), the HN filtrations of Q_1 and $\tilde{Q}_1$ are related by $\mathbb{F}^{\mathrm{hn}}_i(Q_1) = \ker(\mathbb{F}^{\mathrm{hn}}_i(\tilde{Q}_1) \to T)$. For convenience, set $F_i = \mathbb{F}^{\mathrm{hn}}_i(Q_1)$, $\tilde{F}_i = \mathbb{F}^{\mathrm{hn}}_i(\tilde{Q}_1)$. Notice that $\tilde{Q}_1$ and $\hat{Q}_1$ continue to satisfy the hypothesis of the proposition. Hence, by induction, we may assume that for ε sufficiently small the HN filtration of $\hat{Q}_1$ is given by $\mathbb{F}^{\mathrm{hn}}_{i,\varepsilon}(\hat{Q}_1) = \{\mathrm{Sat}_{\hat{Q}_1}(\jmath(\pi^* \tilde{F}_i))\}$ (where $\jmath$ now is the induced map to $\hat{Q}_1$). Now $\pi^* F_i \hookrightarrow \pi^* \tilde{F}_i$ with a torsion quotient. By Lemma 2.1,

$$\mathrm{Sat}_{\hat{Q}_1}(\jmath(\pi^* \tilde{F}_i)) = \mathrm{Sat}_{\hat{Q}_1}(\jmath(\pi^* F_i)).$$

This, combined with Prop. 2.4 (2) gives $\mathbb{F}^{\mathrm{hn}}_{i,\varepsilon}(\hat{E}) = \ker(\hat{E} \to \hat{Q}_1/\mathrm{Sat}_{\hat{Q}_1}(\jmath(\pi^* F_{i-1})))$. Moreover, $F_{i-1} = E_i/E_1$. Clearly then, $\mathbb{F}^{\mathrm{hn}}_{i,\varepsilon}(\hat{E})$ contains $\jmath(\pi^* E_i)$ with a torsion quotient. Thus, again by Lemma 2.1, $\mathbb{F}^{\mathrm{hn}}_{i,\varepsilon}(\hat{E}) = \mathrm{Sat}_{\hat{E}}(\jmath(\pi^* E_i))$. □

Proof of Theorem 3.1. By Prop. 3.7, the HN filtration $\{E_i = \mathbb{F}^{\mathrm{hn}}_{i,\omega}(E)\}$ of E admits a resolution to a filtration $\{\hat{E}_i\}$ by subbundles on $\hat{X}$. By Prop. 3.8, the filtration $\{\hat{E}_i\}$ is the HN filtration with respect to ω_ε for ε sufficiently small. The remaining assertions follow easily. □

3.2. An approximate critical hermitian structure. For a fixed holomorphic structure on $E \to X$, a critical point of the functional $H \mapsto \mathrm{HYM}(\bar{\partial}_E, H)$ is called a critical hermitian structure [Ko], p. 108. By the Kähler identities (2.5), this occurs if and only if the connection $(\bar{\partial}_E, H)$ is Yang-Mills. The general form for a critical hermitian structure is therefore (see Prop. 2.8):

$$\sqrt{-1}\Lambda_\omega F_{(\bar{\partial}_E, H)} = \mu_1 \mathbf{I}_{Q_1} \oplus \cdots \oplus \mu_\ell \mathbf{I}_{Q_\ell}. \tag{3.1}$$

In the above, the holomorphic structure $\bar{\partial}_E$ on E splits $E = \bigoplus_{i=1}^{\ell} Q_i$, and the induced metric on each factor Q_i is Hermitian-Einstein with slope μ_i. Notice that if we assume the slopes are ordered $\mu_1 > \cdots > \mu_\ell$, then the HN filtration of E is given by: $\mathbb{F}^{\mathrm{hn}}_i(E) = \bigoplus_{j \leqq i} Q_j$.

For a general holomorphic structure on E, the HN filtration will not be holomorphically split, so there can exist no smooth metric satisfying (3.1). What is more, the HN filtration may not be given by subbundles, so the right hand side of (3.1) is not even everywhere defined as a smooth endomorphism. In this subsection, we define precisely what is meant by an *approximate* solution to (3.1) (compare the following discussion with that in [Ko], IV. §5).

Let H be a smooth metric on E, and let $\mathscr{F} = \{F_i\}_{i=1}^{\ell}$ be a filtration of E by saturated subsheaves: $\mathscr{F} : 0 = F_0 \subset F_1 \subset \cdots \subset F_\ell = E$. Associated to each F_i and the metric H we have the unitary projection π_i^H onto F_i. As mentioned previously, the π_i^H are bounded L_1^2

hermitian endomorphisms. For convenience, we set $\pi_0^H = 0$. Next, suppose we are given a collection of real numbers $\mu_1, \dots, \mu_\ell$. From the data $\mathscr{F}$ and $(\mu_1, \dots, \mu_\ell)$ we define a bounded L_1^2 hermitian endomorphism of E by $\Psi\big(\mathscr{F}, (\mu_1, \dots, \mu_\ell), H\big) = \sum_{i=1}^{\ell} \mu_i(\pi_i^H - \pi_{i-1}^H)$. At points where the F_i are all subbundles there is a smooth orthogonal splitting: $E = \bigoplus_{i=1}^{\ell} F_i/F_{i-1}$ with respect to which $\Psi\big(\mathscr{F}, (\mu_1, \dots, \mu_\ell), H\big)$ is diagonal with entries μ_i. Given a holomorphic hermitian vector bundle E on a compact Kähler surface (X, ω), the Harder-Narasimhan projection, $\Psi_\omega^{\mathrm{hn}}(\bar\partial_E, H)$, is the bounded L_1^2 hermitian endomorphism defined above in the particular case where $\mathscr{F}$ is the HN filtration $F_i = \mathbb{F}_i^{\mathrm{hn}}(E)$ and $\mu_i = \mu(F_i/F_{i-1})$.

Definition 3.9. Fix $\delta > 0$ and $1 \leqq p \leqq \infty$. An L^p-δ-approximate critical hermitian structure on a holomorphic bundle E is a smooth metric H such that

$$\|\sqrt{-1}\Lambda_\omega F_{(\bar\partial_E, H)} - \Psi_\omega^{\mathrm{hn}}(\bar\partial_E, H)\|_{L^p(\omega)} \leqq \delta.$$

Let us immediately point out the following:

Theorem 3.10. *If the* HN *filtration of E is given by subbundles, then for any $\delta > 0$ there is an L^∞-δ-approximate critical hermitian structure on E.*

Proof. First, by the equivalence of holomorphic structures and integrable unitary connections, it suffices to show that for a *fixed* hermitian metric H there is a smooth complex gauge transformation g preserving the HN filtration such that:

$$\big\|\sqrt{-1}\Lambda_\omega F_{(g(\bar\partial_E), H)} - \Psi_\omega^{\mathrm{hn}}\big(g(\bar\partial_E), H\big)\big\|_{L^\infty(\omega)} \leqq \delta, \tag{3.2}$$

(see [Do1]). Next, for semistable E (i.e. the length 1 case), the result follows by the convergence $\|\sqrt{-1}\Lambda_\omega F_{D_t} - \mu(E)\mathbf{I}_E\|_{L^\infty(\omega)} \to 0$, where D_t is a solution to the Yang-Mills flow equations (2.7) with any initial condition (cf. [Do1], Cor. 25). With this understood, choose δ'-approximate metrics, where $0 < \delta' \ll \delta$, on the semistable quotients Q_i of the HN filtration of E to fix a metric H on $E = Q_1 \oplus \cdots \oplus Q_\ell$. Then by appropriately scaling the extension classes: $0 \to E_{i-1} \to E_i \to Q_i \to 0$, one finds a complex gauge transformation satisfying (3.2). We omit the details. $\square$

We may now formulate one of the main results of this paper:

Theorem 3.11. *Let $E \to (X, \omega)$ be a holomorphic vector bundle on a smooth Kähler surface X. Given any $\delta > 0$ and any $1 \leqq p < \infty$, there is an L^p-δ-approximate critical hermitian structure on E.*

Remark 3.12. The metric produced in Thm. 3.11 depends upon p. In particular, the proof we shall give does not extend to $p = \infty$. This leaves open the following question: Can one find an L^∞-δ-approximate critical hermitian structure in general?

The proof of Thm. 3.11 will be given in Section 4.2 below. A preliminary result in this direction is obtained by passing to a resolution of the filtration. We will prove the following:

Proposition 3.13. *Let E, X, and ω be as in Thm.* 3.11. *Let $\mu_i = \mu_\omega\big(\mathbb{F}_i^{\mathrm{hn}}(E)/\mathbb{F}_{i-1}^{\mathrm{hn}}(E)\big)$. Then there is a sequence of monoidal transformations giving a Kähler surface $\pi : \hat{X} \to X$, a number $p_0 > 1$, and a family of Kähler metrics ω_ε converging to $\pi^*\omega$ as $\varepsilon \to 0$, such that the following holds: Let $\hat{\mathscr{F}}$ be the filtration of $\pi^*E = \hat{E}$ given by $\{\mathrm{Sat}_{\hat{E}}(\pi^*\mathbb{F}_i^{\mathrm{hn}}(E))\}$. Then for any $\delta > 0$ and any $1 \leqq p < p_0$ there is $\varepsilon_1 > 0$ and a smooth hermitian metric $\hat{H}$ on $\hat{E}$ such that for all $0 < \varepsilon \leqq \varepsilon_1$, $\big\|\sqrt{-1}\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\hat{E}}, \hat{H})} - \Psi\big(\hat{\mathscr{F}}, (\mu_1, \ldots, \mu_\ell), \hat{H}\big)\big\|_{L^p(\omega_\varepsilon)} \leqq \delta$.*

For a *fixed* $\varepsilon > 0$ sufficiently small compared to δ, the analogous result for *any* p is a consequence of Thm. 3.10. The key point in the statement of Prop. 3.13 is that a metric $\hat{H}$ may be found which satisfies the condition of the proposition *uniformly* in ε. The direct construction of $\hat{H}$ given below, however, requires that p be sufficiently small. This requirement derives from the following:

Lemma 3.14. *Let $\pi : \hat{X} \to X$ be a blow-up of the type discussed in Section* 3.1, *and let $\omega_\varepsilon = \pi^*\omega + \varepsilon\eta$ be the family of Kähler metrics defined in Lemma* 3.3. *Then there is associated to $\hat{X}$ a positive integer $\hat{m}$ with the following property: given any p, $1 \leqq p < 1 + (1/\hat{m})$, there is $\varepsilon_1 > 0$ such that for any $\tilde{p}$ satisfying $p\big(1 - \hat{m}(p-1)\big)^{-1} < \tilde{p} \leqq +\infty$, there is a constant $C(\tilde{p}, \varepsilon_1)$ such that $\|\Lambda_{\omega_\varepsilon} G\|_{L^p(\omega_\varepsilon)} \leqq C(\tilde{p}, \varepsilon_1)\|\Lambda_{\omega_{\varepsilon_1}} G\|_{L^{\tilde{p}}(\omega_{\varepsilon_1})}$, for all smooth $(1,1)$ forms G on $\hat{X}$ and all $0 < \varepsilon \leqq \varepsilon_1$.*

Proof. Since $\omega_\varepsilon \to \pi^*\omega$ smoothly, and $\pi^*\omega$ is a Kähler metric off the exceptional set, the estimate is clearly local near the exceptional set $\mathbf{e}$. Let $\hat{x} \in \mathbf{e} \subset \hat{X}$ with $x = \pi(\hat{x})$. We may choose local coordinates (z_1, z_2) near x with respect to which the Kähler form ω is standard to first order. Since $\mathbf{e}$ has normal crossings, we may choose coordinates (ξ_1, ξ_2) in a neighborhood $\hat{U}$ of $\hat{x}$, centered at $\hat{x}$, and such that $\mathbf{e} \cap \hat{U}$ is contained in the union of the coordinate axes $\{\xi_1 = 0\} \cup \{\xi_2 = 0\}$. Regarding z_1, z_2 as holomorphic functions on $\hat{U}$, let us write: $z_1 \simeq \xi_1^a \xi_2^m$, $z_2 \simeq \xi_1^b \xi_2^n$, modulo higher order terms, where a, b, m, n are nonnegative integers and $an \neq bm$. In these coordinates we have:

$$\pi^*(\omega \wedge \omega) \simeq |\xi_1|^{2(a+b-1)}|\xi_2|^{2(m+n-1)}(an - bm)^2 \times \left(\frac{\sqrt{-1}}{2} d\xi_1 \wedge d\bar{\xi}_1\right) \wedge \left(\frac{\sqrt{-1}}{2} d\xi_2 \wedge d\bar{\xi}_2\right) \tag{3.3}$$

modulo higher order terms. At this point we set: $\hat{m} = \max\{(a+b-1), (m+n-1)\}$. Let $g_{\alpha\bar{\beta}}^\varepsilon$ denote the Kähler metric with Kähler form ω_ε in the coordinates ξ_α, $\alpha = 1, 2$ on the neighborhood $\hat{U}$. It follows from (3.3) that there is a constant C uniform in ε such that:

$$|\det g_{\alpha\bar{\beta}}^\varepsilon| \geqq C|\xi_1|^{2\hat{m}}|\xi_2|^{2\hat{m}}. \tag{3.4}$$

Let $g_\varepsilon^{\alpha\bar{\beta}} = M_\varepsilon^{\alpha\bar{\beta}}/\det g_{\alpha\bar{\beta}}^\varepsilon$ denote the inverse metric. Since $\dim_{\mathbb{C}} X = 2$, $M_\varepsilon^{\alpha\bar{\beta}} = \epsilon^{\alpha\sigma}\epsilon^{\beta\rho} g_{\sigma\bar{\rho}}^\varepsilon$, where $\epsilon^{11} = \epsilon^{22} = 0$, $\epsilon^{12} = -\epsilon^{21} = 1$. In particular, we can arrange for a constant C uniform in ε such that in a neighborhood of $\hat{x}$ (still denoted $\hat{U}$):

$$|M_\varepsilon^{\alpha\bar{\beta}}| \leqq C|M_{\varepsilon_1}^{\alpha\bar{\beta}}|, \tag{3.5}$$

for all $0 < \varepsilon \leqq \varepsilon_1$. Using (3.4) and (3.5) we now prove the lemma. Let $G_{\alpha\bar{\beta}}$ be the local expression in the coordinates (ξ_1, ξ_2) on $\hat{U}$ of the form G in the statement of the lemma. Then by (3.5) there is a constant C independent of ε and G such that: $|G_{\alpha\bar{\beta}}M_{\varepsilon}^{\alpha\bar{\beta}}|^p \leqq C|G_{\alpha\bar{\beta}}M_{\varepsilon_1}^{\alpha\bar{\beta}}|^p$, for all $p \geqq 1$ and $0 < \varepsilon \leqq \varepsilon_1$. Then:

$$\|\Lambda_{\omega_\varepsilon} G\|^p_{L^p(\hat{U},\omega_\varepsilon)} = \int_{\hat{U}} |G_{\alpha\bar{\beta}}g_\varepsilon^{\alpha\bar{\beta}}|^p (\det g^\varepsilon_{\alpha\bar{\beta}})|d\xi_1|^2|d\xi_2|^2$$

$$\leqq C \int_{\hat{U}} |G_{\alpha\bar{\beta}}g_{\varepsilon_1}^{\alpha\bar{\beta}}|^p \left(\frac{\det g^\varepsilon_{\alpha\bar{\beta}}}{\det g^{\varepsilon_1}_{\alpha\bar{\beta}}}\right)^{1-p} (\det g^{\varepsilon_1}_{\alpha\bar{\beta}})|d\xi_1|^2|d\xi_2|^2.$$

Now $\det g^{\varepsilon_1}_{\alpha\bar{\beta}}$ is uniformly bounded away from zero on $\hat{U}$ by a constant depending upon ε_1. If $\tilde{p} = \infty$, the result follows from (3.4) and the assumption on p. If $\tilde{p} \neq \infty$, we apply Hölder's inequality with the conjugate variables: $r = \tilde{p}/p$, $s = \tilde{p}/(\tilde{p} - p)$, and find:

$$\|\Lambda_{\omega_\varepsilon} G\|^p_{L^p(\hat{U},\omega_\varepsilon)} \leqq C\|\Lambda_{\omega_{\varepsilon_1}} G\|^p_{L^{\tilde{p}}(\hat{U},\omega_{\varepsilon_1})} \left\{ \int_{\hat{U}} (\det g^\varepsilon_{\alpha\bar{\beta}})^{(1-p)s} |d\xi_1|^2|d\xi_2|^2 \right\}^{1/s}. \tag{3.6}$$

From the assumption on $\tilde{p}$ we have $2\hat{m}(1-p)s > -2$. By (3.4), this implies that the integral on the right hand side of (3.6) is convergent uniformly in ε. This proves the lemma. □

Proof of Proposition 3.13. We proceed by induction on the rank of E. The case of rank 1 is trivial, since line bundles admit Hermitian-Einstein metrics. Suppose that $\mathrm{rk}(E) > 1$, and consider the HN filtration $\{\mathbb{F}^{\mathrm{hn}}_{i,\omega}(E)\}$. If the filtration is by subbundles, then the result follows from Thm. 3.10. Consider the case where the filtration is not by subbundles. For convenience, set $E_i = \mathbb{F}^{\mathrm{hn}}_{i,\omega}(E)$, $Q_i = E_i/E_{i-1}$, and $\mu_i = \mu_\omega(Q_i)$. By Prop. 3.7 there is a resolution $\pi : \hat{X} \to X$ where the filtration $\hat{E}_i = \mathrm{Sat}_{\hat{E}}\, \pi^* E$ is a filtration of $\hat{E} = \pi^* E$ by subbundles. Let $\hat{Q}_i = \hat{E}_i/\hat{E}_{i-1}$. By the inductive hypothesis, given $\delta > 0$ and for any $\varepsilon > 0$ sufficiently small we may find $L^{\hat{p}}$ δ-approximate critical hermitian structures $\hat{H}_i^\varepsilon$ on each $\hat{Q}_i$, for some $\hat{p} > 1$. Since E_i/E_{i-1} is semistable, it follows from Prop. 3.4 that for a given δ_1 we may assume ε_1 has been chosen such that $|\mu_{\max,\varepsilon}(\hat{Q}_i) - \mu_{\min,\varepsilon}(\hat{Q}_i)| \leqq \delta_1$ for all $0 < \varepsilon \leqq \varepsilon_1$. Here, $\delta_1 > 0$ will be chosen presently. In particular:

$$\|\Psi^{\mathrm{hn}}_{\omega_\varepsilon}(\bar{\partial}_{\hat{Q}_i}, \hat{H}_i^{\varepsilon_1}) - \mu_i \mathbf{I}_{\hat{Q}_i}\|_{L^{\hat{p}}(\omega_\varepsilon)} \leqq C\delta_1, \tag{3.7}$$

for a constant C independent of ε and δ_1. Associated to $\hat{X}$ is an integer $\hat{m}$ as in Lemma 3.14. We choose p_0 sufficiently close to 1 so that $p_0 < \hat{p}\big(1 - \hat{m}(p_0 - 1)\big)$. Then the conclusion of the lemma, along with (3.7), guarantee that for each $1 \leqq p < p_0$, each i, and each $0 < \varepsilon \leqq \varepsilon_1$, $\|\sqrt{-1}\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\hat{Q}_i}, \hat{H}_i^{\varepsilon_1})} - \mu_i \mathbf{I}_{\hat{Q}_i}\|_{L^p(\omega_\varepsilon)} \leqq C\delta_1$, for a constant C independent of ε and δ_1. Choose a smooth splitting $\hat{E} = \bigoplus_{i=1}^{\ell} \hat{Q}_i$, and let $\hat{H} = \bigoplus_{i=1}^{\ell} \hat{H}_i^{\varepsilon_1}$. This is a smooth metric on $\hat{E}$. Since the filtration $\{\hat{E}_i\}$ is by subbundles, we may argue as in the proof of Thm. 3.10 that $\hat{H}$ may be modified to produce the desired result if we choose δ_1 sufficiently small (depending upon the constant C) compared to δ. □

4. The Harder-Narasimhan type of the Uhlenbeck limit

As indicated in the Introduction, the proof of the Main Theorem proceeds in two steps. The goal of this section is to prove the first step:

Theorem 4.1. *Let D_t be a solution to the* YM *flow equations* (2.7) *with initial condition D_0 and Uhlenbeck limit D_∞. Let E_∞ denote the holomorphic vector bundle obtained from D_∞, as in Prop.* 2.15. *Then the Harder-Narasimhan type of (E_∞, D''_∞) is the same as that of (E, D''_0).*

We prove this theorem in the first subsection below. In the second subsection, we use this fact to prove Thm. 3.11.

4.1. Proof of Theorem 4.1. We begin with the following:

Lemma 4.2. *Let $E \to X$ be a holomorphic bundle of* HN *type $\vec{\mu}_0$. There is $\alpha_0 > 1$ such that the following holds: given any $\delta > 0$ and any N, there is a hermitian metric H on E such that $\mathrm{HYM}_{\alpha,N}(\bar{\partial}_E, H) \leqq \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + \delta$, for all $1 \leqq \alpha \leqq \alpha_0$.*

Proof. To begin, let $\pi : \hat{X} \to X$ be a resolution of the HN filtration guaranteed by Prop. 3.7, ω_ε the family of Kähler metrics from Lemma 3.3, and $\hat{E} = \pi^* E$. As a direct consequence Prop. 3.13, where $\hat{\alpha}_0 = p_0$, and Cor. 3.6, there is $\hat{\alpha}_0 > 1$ such that the following holds: given any $\delta > 0$ there exists a smooth hermitian metric $\hat{H}$ on $\hat{E}$, and $\varepsilon_1 > 0$ (depending on $\hat{H}$) such that:

$$\mathrm{HYM}^{\omega_\varepsilon}_{\alpha,N}(\bar{\partial}_{\hat{E}}, \hat{H}) \leqq \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + \delta/2, \tag{4.1}$$

for all $1 \leqq \alpha \leqq \hat{\alpha}_0$, and all $0 \leqq \varepsilon \leqq \varepsilon_1$. In order to obtain a metric on X, we use a cut-off argument. Let $x \in Z^{\mathrm{alg.}}$, and choose a coordinate neighborhood U of x. For $R > 0$ sufficiently small, let $B_R \subset U$ denote the ball of radius R about x with respect to these coordinates. We also choose a holomorphic trivialization of $E \to U$. This gives a trivialization of $\hat{E}$ on $\hat{U} = \pi^{-1}(U)$, with respect to which we regard $\hat{H}$ as a positive definite hermitian matrix valued function. Given R, we may choose a smooth function φ_R on U, $0 \leqq \varphi_R \leqq 1$, $\varphi_R \equiv 0$ on a ball of radius $R/2$ centered at p, and $\varphi_R \equiv 1$ outside a ball of radius R, and such that $|\varphi'_R| \leqq CR^{-1}$ and $|\varphi''_R| \leqq CR^{-2}$, where C is a constant independent of R. Define a metric H_{φ_R} as follows: If $\hat{H} s_i = \lambda_i s_i$ with respect to a unitary frame $\{s_i\}$, then $H_{\varphi_R} s_i = \big(\varphi_R \lambda_i + (1 - \varphi_R)\big) s_i$. With this definition, H_{φ_R} extends as a smooth metric on $E \to U$. Let $\hat{H}_{\varphi_R}$ denote the pull-back metric on $\hat{X}$. A calculation then shows that there are constants C_1 and C_2, depending on $\hat{H}$ but not on R or ε, such that on $\pi^{-1}(B_R \backslash B_{R/2})$: $\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\hat{E}}, \hat{H}_{\varphi_R})} = \Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\hat{E}}, \hat{H})} + f_{\varepsilon,R}$, where in the coordinates used in the proof of Lemma 3.14:

$$|f_{\varepsilon,R}| \leqq |\det g^\varepsilon_{\alpha\bar{\beta}}|^{-1}(C_1 + C_2 R^{-2}). \tag{4.2}$$

Continuing this way for all points in $Z^{\mathrm{alg.}}$, we obtain a metric, still denoted H_{φ_R}, with $\hat{H}_{\varphi_R} = \hat{H}$ outside the union U_R of the balls B_R, $\mathrm{vol}_\omega(U_R) \simeq R^4$, and H_{φ_R} standard with respect to the trivialization inside $U_{R/2}$. Hence:

$$\begin{aligned} &|\mathrm{HYM}^{\omega_\varepsilon}_{\alpha,N}(\bar{\partial}_{\hat{E}}, \hat{H}_{\varphi_R}) - \mathrm{HYM}^{\omega_\varepsilon}_{\alpha,N}(\bar{\partial}_{\hat{E}}, \hat{H})| \\ &\qquad \leqq CR^4 + C\|\Lambda_{\omega_\varepsilon} F_{(\bar{\partial}_{\hat{E}}, \hat{H})}\|^\alpha_{L^\alpha_{\omega_\varepsilon}(\pi^{-1}(U_R))} + C\|f_{\varepsilon,R}\|^\alpha_{L^\alpha_{\omega_\varepsilon}(\pi^{-1}(U_R \backslash U_{R/2}))} \end{aligned}$$

where C is independent of R and ε. By the construction of $\hat{H}$, the second term on the right hand side tends to zero as $R \to 0$, uniformly in ε. Hence, we may choose R sufficiently small so that this term is less than $\delta/4$, say, for all $\varepsilon \leqq \varepsilon_1$. Letting $\varepsilon \to 0$ and using (4.2) to bound the third term, we obtain an estimate of the form:

$$|\mathrm{HYM}^{\omega}_{\alpha,N}(\bar\partial_E, H_{\varphi_R}) - \mathrm{HYM}^{\omega_0}_{\alpha,N}(\bar\partial_{\hat E}, \hat H)| \leqq C(1 + R^{-2\alpha})R^4 + \delta/4.$$

Now by (4.1), provided $\alpha_0 \leqq \hat\alpha_0$ and $\alpha_0 < 2$, we may take R sufficiently small so that:

$$\mathrm{HYM}^{\omega}_{\alpha,N}(\bar\partial_E, H_{\varphi_R}) \leqq \mathrm{HYM}_{\alpha,N}(\vec\mu_0) + \delta$$

for all $\alpha \leqq \alpha_0$. $\square$

Next, we have the following "distance decreasing" result:

Lemma 4.3. *Let α_0 be as in Lemma 4.2. Let H be any smooth hermitian metric on E, and let D_t be a solution to the* YM *flow equations* (2.7) *with initial condition* $(\bar\partial_E, H)$. *Then*:

$$\lim_{t\to\infty} \mathrm{HYM}_{\alpha,N}(D_t) = \mathrm{HYM}_{\alpha,N}(\vec\mu_0),$$

for all $1 \leqq \alpha \leqq \alpha_0$, *and all N. In particular, if D_∞ is the Uhlenbeck limit along the flow, then* $\mathrm{HYM}_{\alpha,N}(D_\infty) = \mathrm{HYM}_{\alpha,N}(\vec\mu_0)$.

Proof. We first point out that the second assertion follows from the first because of Prop. 2.26. For fixed α, $1 \leqq \alpha \leqq \alpha_0$, and fixed N, define $\delta_0 > 0$ by:

$$(4.3)\qquad 2\delta_0 + \mathrm{HYM}_{\alpha,N}(\vec\mu_0) = \min\{\mathrm{HYM}_{\alpha,N}(\vec\mu) : \mathrm{HYM}_{\alpha,N}(\vec\mu) > \mathrm{HYM}_{\alpha,N}(\vec\mu_0)\},$$

where $\vec\mu$ runs over all possible HN types of holomorphic vector bundles on X with the rank of E. Consider metrics H on E with associated connection $D = (\bar\partial_E, H)$ satisfying:

$$(4.4)\qquad \mathrm{HYM}_{\alpha,N}(D) \leqq \mathrm{HYM}_{\alpha,N}(\vec\mu_0) + \delta_0.$$

Let D_∞ be the Uhlenbeck limit along the flow with initial condition D. Then combining Prop. 2.21, Prop. 2.24 (1), and Prop. 2.25, we have:

$$\mathrm{HYM}_{\alpha,N}(\vec\mu_0) \leqq \mathrm{HYM}_{\alpha,N}(D_\infty) \leqq \mathrm{HYM}_{\alpha,N}(D) \leqq \mathrm{HYM}_{\alpha,N}(\vec\mu_0) + \delta_0.$$

Hence, by (4.3) we must have $\mathrm{HYM}_{\alpha,N}(D_\infty) = \mathrm{HYM}_{\alpha,N}(\vec\mu_0)$. This shows that the result holds for initial conditions satisfying (4.4).

In the following, let us denote by D_t^H the solution to the YM flow at time t with initial condition $D_0^H = (\bar\partial_E, H)$. We are going to prove that for *any* initial condition H and any $\delta > 0$, there is $T \geqq 0$ such that:

$$(4.5)\qquad \mathrm{HYM}_{\alpha,N}(D_t^H) < \mathrm{HYM}_{\alpha,N}(\vec\mu_0) + \delta, \quad \text{for all } t \geqq T.$$

Without loss of generality, assume $0 < \delta \leqq \delta_0/2$. Let $\mathscr{H}_\delta$ denote the set of smooth hermitian metrics H on E with the property that (4.5) holds for D_t^H and some T. From the discussion above, $\mathscr{H}_\delta$ is nonempty: indeed, any metric satisfying (4.4) is in $\mathscr{H}_\delta$, and according to Lemma 4.2 we may always find such a metric. Let H^j be a sequence of smooth hermitian metrics on E such that each $H^j \in \mathscr{H}_\delta$, and suppose $H^j \to K$, in the C^∞ topology, for some metric K. Since $H^j \in \mathscr{H}_\delta$ we have a sequence T_j such that for all $t \geqq T_j$:

$$\mathrm{HYM}_{\alpha,N}(D_t^{H^j}) \leqq \mathrm{HYM}_{\alpha,N}(D_{T_j}^{H^j}) \leqq \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + \delta. \tag{4.6}$$

By Lemma 2.14 (2) and the C^∞ convergence of H^j, $\|\Lambda F_{D_t^{H^j}}\|_{L^\infty}$ and $\|\Lambda F_{D_t^K}\|_{L^\infty}$ are bounded uniformly for all $t \geqq 0$. Hence, it follows from Prop. 2.11 that we may find a sequence $t_j \geqq T_j$, Yang-Mills connections $D_\infty^{(1)}$ and $D_\infty^{(2)}$, and bubbling sets $Z_{(1)}^{\mathrm{an.}}$ and $Z_{(1)}^{\mathrm{an.}}$, such that $D_{t_j}^{H^j} \rightharpoonup D_\infty^{(1)}$ in $L^p_{1,\mathrm{loc.}}(X \backslash Z_{(1)}^{\mathrm{an.}})$ and $D_{t_j}^K \rightharpoonup D_\infty^{(2)}$ in $L^p_{1,\mathrm{loc.}}(X \backslash Z_{(2)}^{\mathrm{an.}})$, for all $1 \leqq p < \infty$. Moreover, by Cor. 2.12, $\Lambda F_{D_{t_j}^{H^j}} \to \Lambda F_{D_\infty^{(1)}}$ and $\Lambda F_{D_{t_j}^K} \to \Lambda F_{D_\infty^{(2)}}$ strongly in L^p, for all p.

Claim. $D_\infty^{(1)} = D_\infty^{(2)}$.

Proof of the Claim. Write $H_{t_j}^j = h_{t_j}^j K_{t_j}$. It follows by [Do1], Prop. 13, that $\sup \sigma(H_t^j, K_t) \to 0$ as $j \to \infty$, uniformly in t, where:

$$\sigma(H,K) = \operatorname{Tr} H^{-1}K + \operatorname{Tr} K^{-1}H - 2\operatorname{rk}(E),$$

is the usual C^0-distance on the space of hermitian metrics on E. In particular, $\sup|h_{t_j}^j - \mathbf{I}_E| \to 0$ as $j \to \infty$. Let $Z^{\mathrm{an.}} = Z_{(1)}^{\mathrm{an.}} \cup Z_{(2)}^{\mathrm{an.}}$, and choose a smooth test form $\phi \in \Omega^{1,0}(\operatorname{End} E)$, compactly supported on $X \backslash Z^{\mathrm{an.}}$. We have

$$(D_{t_j}^{H^j})' - (D_{t_j}^K)' = (h_{t_j}^j)^{-1}(D_{t_j}^K)'(h_{t_j}^j).$$

For notational simplicity, set $D_j = D_{t_j}^K$, and $h_j = h_{t_j}^j$. Then there is a constant C such that

$$|\langle h_j^{-1}D_j'(h_j), \phi\rangle_{L^2}| \leqq C|\langle h_j, (D_j')^*\phi\rangle_{L^2}| \leqq C\{|\langle h_j, (D_j' - D_\infty')^*\phi\rangle_{L^2}| + |\langle h_j, (D_\infty')^*\phi\rangle_{L^2}|\}.$$

Now $D_j' \rightharpoonup D_\infty'$ in $L^p_{1,\mathrm{loc.}}$, so we may assume $D_j' \to D_\infty'$ in C^0. Combined with the uniform bound for $\|h_j\|_{L^\infty}$, this implies that the first term on the right hand side above goes to zero. For the second term, notice that since $h_j \xrightarrow{C^0} \mathbf{I}_E$:

$$\langle h_j, (D_\infty')^*\phi\rangle_{L^2} \to \langle \mathbf{I}_E, (D_\infty')^*\phi\rangle_{L^2} = \int_X \operatorname{Tr}(D_\infty')^*\phi\, dvol = \int_X \partial^* \operatorname{Tr}\phi\, dvol = 0,$$

by Stokes' theorem. This proves that $(D_{t_j}^{H^j} - D_{t_j}^K) \rightharpoonup 0$ in $L^2_{\mathrm{loc.}}(X \backslash Z^{\mathrm{an.}})$, and the claim follows. □

Set $D_\infty = D_\infty^{(1)} = D_\infty^{(2)}$. Since $\Lambda F_{D_{t_j}^{H^j}} \to \Lambda F_{D_\infty}$ and $\Lambda F_{D_{t_j}^K} \to \Lambda F_{D_\infty}$ strongly in L^p, for all $1 \leqq p < \infty$, we have (see Lemma 2.23):

$$\lim_{j\to\infty} \mathrm{HYM}_{\alpha,N}(D_{t_j}^{H^j}) = \lim_{j\to\infty} \mathrm{HYM}_{\alpha,N}(D_{t_j}^K) = \mathrm{HYM}_{\alpha,N}(D_\infty).$$

Hence, for j sufficiently large:

$$\begin{aligned}\mathrm{HYM}_{\alpha,N}(D_{t_j}^K) &\leqq \mathrm{HYM}_{\alpha,N}(D_\infty) + \delta = \lim_{j\to\infty} \mathrm{HYM}_\alpha(D_{t_j}^{H^j}) + \delta \\ &\leqq \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + 2\delta \leqq \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + \delta_0,\end{aligned}$$

where in the second line we have used (4.6) and the fact that $\delta \leqq \delta_0/2$. It follows from (4.3) and the discussion above that for j sufficiently large: $\lim_{t\to\infty} \mathrm{HYM}_{\alpha,N}(D^K_{t_j+t}) = \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0)$. In particular, $\mathrm{HYM}_{\alpha,N}(D^K_{t_j+t}) < \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0) + \delta$, for $t \geqq T$, T sufficiently large. Therefore, $K \in \mathscr{H}_\delta$. This proves that $\mathscr{H}_\delta$ is closed in the C^∞ topology. By the continuous dependence of the flow on initial conditions, $\mathscr{H}_\delta$ is also open. Since the space of smooth metrics is connected, we conclude that every metric is in $\mathscr{H}_\delta$, and (4.5) holds for all $\delta > 0$ and all initial conditions H. In particular, we can choose $\delta \leqq \delta_0$ and conclude that $\lim_{t\to\infty} \mathrm{HYM}_{\alpha,N}(D^H_t) = \mathrm{HYM}_{\alpha,N}(\vec{\mu}_0)$, for any H. Since the choice of N was arbitrary, the proof is complete. $\square$

Finally, we have:

Proof of Theorem 4.1. Let $\vec{\mu}_0 = (\mu_1, \ldots, \mu_R)$ (resp. $\vec{\lambda}_\infty = (\lambda_1, \ldots, \lambda_R)$) be the HN type of (E, D''_0) (resp. (E, D''_∞)). By Lemma 4.3, $\varphi_\alpha(\vec{\mu}_0 + N) = \varphi_\alpha(\vec{\lambda}_\infty + N)$ for all $1 \leqq \alpha \leqq \alpha_0$ and all N. In particular, we may choose N sufficiently large so that $\mu_R + N \geqq 0$. By Prop. 2.21 we also have $\lambda_R + N \geqq 0$. Since $\alpha_0 > 0$, the hypotheses of Prop. 2.24 (2) are then satisfied, and we conclude that $\vec{\mu}_0 + N = \vec{\lambda}_\infty + N$, and so $\vec{\mu}_0 = \vec{\lambda}_\infty$. $\square$

4.2. Proof of Theorem 3.11. Let (E, D''_0) be a holomorphic bundle, $D_j = g_j(D_0)$ a sequence of unitary connections in the $\mathfrak{G}^{\mathbb{C}}$ orbit of D_0, and set $F_j = F_{D_j}$. For the next result we make the following assumptions:

Assumptions 4.4. (1) D_j converges off a finite set of points $Z^{\mathrm{an.}} \subset X$ weakly in $L^p_{1,\mathrm{loc.}}$, for all $p > 4$, to a Yang-Mills connection D_∞ on a bundle E_∞.

(2) The HN type of (E_∞, D''_∞) is the same as the HN type of (E, D''_0).

(3) $\|\Lambda F_j\|_{L^\infty}$ is bounded uniformly in j, and $\Lambda F_j \xrightarrow{L^1} \Lambda F_\infty$, where $F_\infty = F_{D_\infty}$.

Recall that for a weakly holomorphic projection π of E, the rank and degree of π are, by definition, the rank and degree of the associated saturated subsheaf of E (see the discussion in the proof of Lemma 2.20).

Lemma 4.5. (1) *Let $\{\pi^{(i)}_j\}$ be the* HN *filtration of (E, D''_j) and $\{\pi^{(i)}_\infty\}$ the* HN *filtration of (E_∞, D''_∞). Then after passing to a subsequence, $\pi^{(i)}_j \to \pi^{(i)}_\infty$ strongly in $L^p \cap L^2_{1,\mathrm{loc.}}$, for all $1 \leqq p < \infty$ and all i.*

(2) *Suppose (E, D''_0) is semistable and $\{\pi^{(i)}_{\mathrm{ss},j}\}$ are Seshadri filtrations of (E, D''_j). Without loss of generality, assume the ranks of the subsheaves $\pi^{(i)}_{\mathrm{ss},j}$ are constant in j. Then there is a filtration $\{\pi^{(i)}_{\mathrm{ss},\infty}\}$ of (E_∞, D''_∞) such that after passing to a subsequence, $\pi^{(i)}_{\mathrm{ss},j} \to \pi^{(i)}_{\mathrm{ss},\infty}$ strongly in $L^p \cap L^2_{1,\mathrm{loc.}}$, for all $1 \leqq p < \infty$ and all i. Moreover, the rank and degree of $\pi^{(i)}_{\mathrm{ss},\infty}$ is equal to the rank and degree of $\pi^{(i)}_{\mathrm{ss},j}$ for all i and j.*

Proof. For part (1), set $E_i = \mathbb{F}^{\mathrm{hn}}_i(E, D''_0)$ and $E^{(i)}_\infty = \mathbb{F}^{\mathrm{hn}}_i(E_\infty, D''_\infty)$. Hence, $\pi^{(i)}_j$ is the orthogonal projection onto the subsheaf $g_j(E_i)$ of (E, D''_j). As in the proof of Lemma 2.20 we have:

$$\deg(E_i) + \frac{1}{2\pi} \int_X \|D''_j \pi^{(i)}_j\|^2 \, dvol \leqq \sum_{k \leqq \mathrm{rk}(E_i)} \mu_k + \frac{1}{2\pi} \|\Lambda F_j - \Lambda F_\infty\|_{L^1},$$

where $\vec{\mu} = (\mu_1, \dots, \mu_R)$ is the HN type of (E_∞, D''_∞). By the assumption 4.4 (2), $\vec{\mu}$ is also the HN type of (E, D''_0), so $\deg(E_i) = \sum_{k \leqq \mathrm{rk}(E_i)} \mu_k$. It then follows from 4.4 (3) that:

$$D''_j \pi_j^{(i)} \to 0 \quad \text{in } L^2. \tag{4.7}$$

Write

$$D''_\infty \pi_j^{(i)} = D''_j \pi_j^{(i)} + (D''_\infty - D''_j)\pi_j^{(i)}. \tag{4.8}$$

We may assume, after perhaps passing to a subsequence, that $\pi_j^{(i)} \rightharpoonup \tilde{\pi}_\infty^{(i)}$ in $L^2_{1,\mathrm{loc.}}$, for some L^2_1 projection $\tilde{\pi}_\infty^{(i)}$. Since $\pi_j^{(i)}$ is uniformly bounded, $\pi_j^{(i)} \xrightarrow{L^p} \tilde{\pi}_\infty^{(i)}$ for all p. Then as in the proof of Prop. 2.11 we conclude from (4.7) and (4.8) that $D''_\infty \tilde{\pi}_\infty^{(i)} = 0$. In particular, $\tilde{\pi}_\infty^{(i)}$ defines a saturated subsheaf $\tilde{E}_\infty^{(i)}$ of E_∞. Furthermore, it is clear that $\mathrm{rk}(\tilde{E}_\infty^{(i)}) = \mathrm{rk}(E_\infty^{(i)})$. Also, we claim that $\deg(\tilde{E}_\infty^{(i)}) = \deg(E_\infty^{(i)})$. To see this, note that since $D''_\infty \tilde{\pi}_\infty^{(i)} = 0$, and $\Lambda F_j \to \Lambda F_\infty$ and $\pi_j^{(i)} \to \tilde{\pi}_\infty^{(i)}$ in L^2:

$$\begin{aligned}
\deg(\tilde{E}_\infty^{(i)}) &= \frac{1}{2\pi} \int_X \mathrm{Tr}(\sqrt{-1}\Lambda F_\infty \tilde{\pi}_\infty^{(i)})\, dvol = \lim_{j\to\infty} \frac{1}{2\pi} \int_X \mathrm{Tr}(\sqrt{-1}\Lambda F_j \pi_j^{(i)})\, dvol \\
&= \deg(E_\infty^{(i)}) + \frac{1}{2\pi} \lim_{j\to\infty} \|D''_j \pi_j^{(i)}\|^2_{L^2} = \deg(E_\infty^{(i)}),
\end{aligned}$$

as claimed. Now the maximal destabilizing subsheaf $\mathbb{F}_1^{\mathrm{hn}}(E_\infty)$ of E_∞ is the *unique* saturated subsheaf of E_∞ with this rank and slope (cf. [Ko], Lemma V.7.17). Hence, $\tilde{\pi}_\infty^{(1)} = \pi_\infty^{(1)}$. Notice also that since $D''_j \xrightarrow{L^2_{\mathrm{loc.}}} D''_\infty$, (4.7) and (4.8) imply that $\pi_j^{(1)} \xrightarrow{L^2_{1,\mathrm{loc.}}} \pi_\infty^{(1)}$. Proceed by induction as follows: fix $1 \leqq k < \ell$, and assume $\tilde{\pi}_\infty^{(i)} = \pi_\infty^{(i)}$ for $i \leqq k$. Then $\tilde{E}_\infty^{(k+1)}/E_\infty^{(k)}$ has the same rank and slope as the maximal destabilizing subsheaf of $E_\infty/E_\infty^{(k)}$, and is therefore equal to it as above. Again we conclude that $\tilde{E}_\infty^{(k+1)} = E_\infty^{(k+1)}$. Continuing until $k = \ell$ completes the proof of part (1) of the lemma.

For part (2), notice that the argument given above applies to a sequence of Seshadri filtrations as well, where because of the lack of uniqueness of Seshadri filtrations we may conclude only that the ranks and degrees of $\tilde{E}_\infty^{(i)}$ correspond to those of $E_\infty^{(i)}$. $\square$

Proof of Theorem 3.11. Let D_t denote a solution to the YM flow equations on $E \to X$ with initial condition $D_0 = (\bar{\partial}_E, H)$, and let D_∞ be the Uhlenbeck limit for some sequence D_{t_j}. Then $\|F_{D_{t_j}}\|_{L^\infty}$ is uniformly bounded, and by Lemma 2.17, $\Lambda F_{D_{t_j}} \xrightarrow{L^p} \Lambda F_{D_\infty}$ for all $1 \leqq p < \infty$. Moreover, we have shown in Thm. 4.1 that the HN type of the limit (E_∞, D''_∞) is the same as that of (E, D''_0). Hence, the assumptions 4.4 (1)–(3) are satisfied, and we may apply Lemma 4.5 to conclude that $\Psi_j \xrightarrow{L^p} \Psi_\infty$ for all p, where $\Psi_j = \Psi_\omega^{\mathrm{hn}}(D''_{t_j}, H)$ and $\Psi_\infty = \Psi_\omega^{\mathrm{hn}}(D''_\infty, H_\infty)$. Since D_∞ is Yang-Mills, $\sqrt{-1}\Lambda F_{D_\infty} = \Psi_\infty$ (cf. Prop. 2.8). So $\|\sqrt{-1}\Lambda F_{D_{t_j}} - \Psi_j\|_{L^p} \leqq \|\Lambda F_{D_{t_j}} - \Lambda F_{D_\infty}\|_{L^p} + \|\Psi_j - \Psi_\infty\|_{L^p} \to 0$. $\square$

5. Proof of the Main Theorem

In this final section we complete the proof of the Main Theorem. The missing ingredient is an identification of the holomorphic structure of the Uhlenbeck limit. As stated in the Introduction, this part of the argument applies to minimizing sequences as well. Therefore, both Thm.'s 1 and 2 will follow from:

Theorem 5.1. *Let D_0 be an integrable unitary connection on $E \to X$, and let $\vec{\mu}_0$ be the* HN *type of (E, D_0''). Suppose that D_j is a sequence of integrable connections in the complex gauge orbit of D_0 such that*: $\mathrm{HYM}(D_j) \to \mathrm{HYM}(\vec{\mu}_0)$. *Then there is a* YM *connection D_∞ on a hermitian bundle $E_\infty \to X$ and a finite set of points $Z^{\mathrm{an.}}$ such that*:

(1) *(E_∞, D_∞'') is holomorphically isomorphic to* $\mathrm{Gr}^{\mathrm{hns}}(E, D_0'')^{**}$.

(2) *E and E_∞ are identified outside $Z^{\mathrm{an.}}$ via $L^p_{2,\mathrm{loc.}}$ isometries for all p.*

(3) *Via the isometries in* (2), *and after passing to a subsequence, $D_j \to D_\infty$ in $L^2_{\mathrm{loc.}}$ away from $Z^{\mathrm{an.}}$.*

The main idea for the proof of Thm. 5.1 follows Donaldson [Do1] who constructs a nontrivial holomorphic map $(E, D_0'') \to (E_\infty, D_\infty'')$. With such a map in hand, one may then apply the basic principle that a nontrivial holomorphic map between stable bundles of the same rank and degree must be an isomorphism.

First, however, let us reduce the problem to the case where the Hermitian-Einstein tensors ΛF_{D_j} are uniformly bounded. Let $D_{j,t}$ denote the solution to the YM flow equations with initial conditions D_j at time t. Fix $t_0 > 0$. It follows from Lemma 2.14 that

$$|\Lambda F_{D_{j,t}}|^2(x) \leqq \int_X K_t(x,y) |\Lambda F_{D_{j,t}}|^2(y)\, dvol(y),$$

where $K_t(x,y)$ is the heat kernel on X. Since $0 < K_t(x,y) \leqq C(1 + t^{-2})$ for some constant C (cf. [CL]), it follows that for $t \geqq t_0 > 0$, $\|\Lambda F_{D_{j,t}}\|_{L^\infty}$ is uniformly bounded for all j in terms of $\|\Lambda F_{D_j}\|_{L^2}$. Note also that $\mathrm{HYM}(\vec{\mu}_0) \leqq \mathrm{HYM}(D_{j,t}) \leqq \mathrm{HYM}(D_{j,t_0}) \leqq \mathrm{HYM}(D_j)$. Next, fix $\delta_0 > 0$. By Prop.'s 2.25, 2.26, and Thm. 4.1, it follows that for each j we may find $t_j \geqq t_0$ such that:

$$\mathrm{HYM}_\alpha(\vec{\mu}_0) \leqq \mathrm{HYM}_\alpha(D_{j,t_j}) \leqq \mathrm{HYM}_\alpha(\vec{\mu}_0) + \delta_0, \tag{5.1}$$

for all $1 \leqq \alpha \leqq 2$. Moreover, as in the proof of Prop. 2.15, we may choose the $\{t_j\}$ so that $\|D_{j,t_j} \Lambda F_{D_{j,t_j}}\|_{L^2} \to 0$. By Prop. 2.11, we may assume, after passing to a subsequence, that D_{j,t_j} has an Uhlenbeck limit D_∞ which is a Yang-Mills connection on a bundle E_∞, $L^p_{2,\mathrm{loc.}}$ isometric to E off a finite set of points $Z^{\mathrm{an.}}$. Moreover, if δ_0 is chosen sufficiently small in (5.1), then the HN type of (E_∞, D_∞'') is $\vec{\mu}_0$ (see Section 4.1). We now argue as in the proof of Prop. 2.18 (see also (2.4)):

$$2\|D_{j,t_j} - D_j\|^2_{L^2} \leqq \mathrm{HYM}(D_j) - \mathrm{HYM}(D_{j,t_j}) \leqq \mathrm{HYM}(D_j) - \mathrm{HYM}(\vec{\mu}_0).$$

Since $D_{j,t_j} \xrightarrow{L^p_{\mathrm{loc.}}} D_\infty$ and $\mathrm{HYM}(D_j) \to \mathrm{HYM}(\vec{\mu}_0)$, it follows that $D_j \xrightarrow{L^2_{\mathrm{loc.}}} D_\infty$. Therefore, we may assume from the beginning that $\|\Lambda F_{D_j}\|_{L^\infty}$ is bounded uniformly in j.

Let $Z^{\mathrm{an.}}$ denote the bubbling set of Uhlenbeck limit $D_j \rightharpoonup D_\infty$. Associated to an initial HNS filtration $\{\pi_0^{(i)}\}$ of (E, D_0'') there is an algebraic singular set $Z^{\mathrm{alg.}}$. Set

$Z = Z^{\mathrm{an.}} \cup Z^{\mathrm{alg.}}$ and $\Omega = X \backslash Z$. Next, we recall from Lemma 4.5 that we may assume there is a sequence $\{\pi_j^{(i)}\}$ of HNS filtrations of (E, D_j''), with ranks constant in j, such that for each i, $\pi_j^{(i)} \to \pi_\infty^{(i)}$ in $L^p \cap L^2_{1,\mathrm{loc.}}(\Omega)$. Here, $\{\pi_\infty^{(i)}\}$ is a filtration of (E, D_∞'') on X by holomorphic subbundles with the same ranks and degrees as the $\pi_j^{(i)}$. We will prove the result inductively on the length of the HNS filtration. The inductive hypotheses on the bundle $E \to \Omega$ are the following:

Hypotheses 5.2. (1) $D_j = g_j(D_0)$ on Ω for complex gauge transformations g_j.

(2) $D_j \rightharpoonup D_\infty$ weakly in $L^p_{1,\mathrm{loc.}}(\Omega)$, where D_∞ is Yang-Mills.

(3) (E, D_0'') and (E_∞, D_∞'') extend to X as reflexive sheaves with the same HN type.

(4) ΛF_{D_j} is bounded in $L^\infty_{\mathrm{loc.}}(\Omega)$ uniformly in j.

The conclusion of the inductive argument will be that (E_∞, D_∞'') is holomorphically isomorphic to $\mathrm{Gr}^{\mathrm{hns}}(E, D_0'')^{**}$.

To achieve this, let $S \subset (E, D_0'')$ be the stable subbundle with $\mu(S) = \mu_{\max}(E, D_0'')$ corresponding to the initial element $\pi_0 = \pi_0^{(1)}$ of the filtration $\{\pi_0^{(i)}\}$, and let $Q = E/S$. It follows from Prop. 2.4 (3) that $\mathrm{Gr}^{\mathrm{hns}}(E, D_0'') = S \oplus \mathrm{Gr}^{\mathrm{hns}}(Q)$. Let $\Omega_0 \subset \Omega$ be the complement of a union of balls around the points of Z. By the proof of [Bu3], Lemma 2.2 (which also works for weak L_1^p convergence, $p > 4$; see also Prop. 2.10), after passing to a subsequence we may find holomorphic maps $f_j : S \to (E, D_j'')$ which converge weakly in $L^p_{2,\mathrm{loc.}}(\Omega)$ to a nonzero holomorphic map $f_\infty : S \to (E, D_\infty'')$. The map f_∞, in turn, extends to X by Hartog's Theorem. If π_j denotes the projection to $f_j(S)$, then as mentioned above $\pi_j \xrightarrow{L^2_{1,\mathrm{loc.}}} \pi_\infty$, where π_∞ is a subbundle of the same rank and degree as S, and $\pi_\infty f_\infty = f_\infty$, $D_\infty'' \pi_\infty = 0$. Write $\mathrm{Gr}^{\mathrm{hns}}(E, D_\infty'') = S_\infty \oplus Q_\infty$, where $S_\infty = \pi_\infty(E)$, $Q_\infty = \ker \pi_\infty$. It follows (cf. [Ko], Cor. V.7.12) that f_∞ must be an isomorphism onto its image: $S \to S_\infty \subset E$. In particular, f_∞ is everywhere injective. Since $f_j \to f_\infty$ locally uniformly on Ω, it is easy to verify that $\pi_j \to \pi_\infty$ locally uniformly as well. By Lemma 2.13 we may assume, after passing to a subsequence, that $\pi_j \to \pi_\infty$ weakly in $L^p_{2,\mathrm{loc.}}(\Omega)$ and strongly in $L^p_{1,\mathrm{loc.}}(\Omega)$ for all p. After applying a suitable sequence of gauge transformations which are uniformly bounded in $L^p_{1,\mathrm{loc.}}(\Omega)$ (cf. [D], Lemma 5.12) we may assume from the beginning that D_j'' *preserves* the subbundle S. With this understood, we are ready to use induction. We have shown that the induced connection $\pi_0 D_j \pi_0$ on S converges to a connection on S_∞ whose holomorphic structure is isomorphic to S. This is the first step in the induction. Now consider the induced connections $D_j^Q = \pi_0^\perp D_j \pi_0^\perp$ on Q. These still satisfy the hypotheses 5.2 (1)–(3) above. By Lemma 2.13, the second fundamental forms $D_j'' \pi_0$ are locally uniformly bounded, so by [Ko], I.6.12, D_j^Q also satisfy 5.2 (4). By induction then, $Q_\infty \simeq \mathrm{Gr}^{\mathrm{hns}}(Q, (D_0^Q)'')^{**}$. This completes the proof of Thm. 5.1.

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