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Factorization of rank two theta functions

II. Proof of the Verlinde formula

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Contents

1 Introduction

In this paper we show how the arguments of [DW3] may be extended to prove a general result concerning the factorization of holomorphic sections of line bundles on the moduli spaces of rank two parabolic bundles on Riemann surfaces. This is the inductive step needed to prove the celebrated Verlinde formula, which may be described as follows: Let k be a positive integer, and define

$$S_{mn} = \left(\frac{2}{k+2}\right)^{1/2} \sin\frac{(m+1)(n+1)\pi}{k+2}. \tag{1.1}$$

For integers g (thought of as a genus) and integers $n_1, \ldots, n_m$, set

$$N^{g,k}_{n_1,\ldots,n_m} = \sum_{l=0}^{k} (S_{l0})^{2-2g} \frac{S_{ln_1}}{S_{l0}} \cdots \frac{S_{ln_m}}{S_{l0}}, \tag{1.2}$$

if each $0 \leqq n_j \leqq k$, and set $N^{g,k}_{n_1,\ldots,n_m} = 0$ otherwise. Now pick a compact Riemann surface Σ of genus g and distinct points $p_1, \ldots, p_m$. Let $\mathscr{A}$ denote the affine space of holomorphic structures on a trivial rank 2 hermitian bundle

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$E \to \Sigma$ with a fixed isomorphism class on $\bigwedge^2 E$, let $\mathcal{G}$ be the group of unitary automorphisms of E with determinant 1, and define the "multiplicity space"

$$\mathcal{V}^{g,k}_{n_1,\dots,n_m} = \mathrm{Hom}_{\mathcal{G}}(V_{n_1} \otimes \cdots \otimes V_{n_m}, H^0(\mathcal{A}, \Delta^{\otimes k})) . \tag{1.3}$$

Here, V_{n_j} denotes the $n_j + 1$ dimensional irreducible representation of SU(2) on which $\mathcal{G}$ acts by evaluation at p_j, Δ is the determinant line bundle $\det \bar{\partial}_E$, and the homomorphisms in (1.3) are required to intertwine the action of $\mathcal{G}$. Finally, for $n_1, \dots, n_m \geqq 0$, let $D^{g,k}_{n_1,\dots,n_m} = \dim \mathcal{V}^{g,k}_{n_1,\dots,n_m}$, and set $D^{g,k}_{n_1,\dots,n_m} = 0$ otherwise. Then we prove

Theorem 1.4. (Verlinde formula for SU(2)) *For $g \geqq 2$ and all integers $n_1, \dots, n_m$, $D^{g,k}_{n_1,\dots,n_m} = N^{g,k}_{n_1,\dots,n_m}$.*

Implicit in this statement is the assertion that the dimensions $D^{g,k}_{n_1,\dots,n_m}$ are independent of the choice of complex structure on Σ (see Theorem 3.8).

There are two special cases of Theorem 1.4 that are worth noting: First, let $\mathcal{M}_s = \mathcal{A}_s/\mathcal{G}^{\mathbb{C}}$ denote the moduli space of stable rank two bundles and trivial determinant on Σ. Then $\Delta \to \mathcal{M}_s$ extends to the semi-stable compactification $\bar{\mathcal{M}}$ as the ample generator Θ_0 of $\mathrm{Pic}(\bar{\mathcal{M}})$. The holomorphic sections of $\Theta_0 \to \mathcal{M}_s$ also extend uniquely (see [DN]). Then Theorem 1.4 reduces to a statement about the dimension of this space of sections:

$$\dim H^0(\bar{\mathcal{M}}, \Theta_0^{\otimes k}) = \left(\frac{k+2}{2}\right)^{1/2} \sum_{l=0}^{k} \left(\sin \frac{(l+1)\pi}{k+2}\right)^{2-2g} . \tag{1.5}$$

Second, let $\mathcal{M}(\mathcal{O}(-p))$ denote the moduli space of stable rank two bundles and determinant $\simeq \mathcal{O}(-p)$. Then $\mathcal{M}(\mathcal{O}(-p))$ also has an ample generator Θ_1, and as explained in [DW2] the dimension of its space of sections is computed by the case $m = 1$, $n_1 = k$ of Theorem 1.4 with the puncture at p. Explicitly, for k even we have

$$\begin{aligned} &\dim H^0(\mathcal{M}(\mathcal{O}(-p)), \Theta_1^{\otimes k/2}) \\ &= \left(\frac{k+2}{2}\right)^{1/2} \sum_{l=0}^{k} (-1)^l \left(\sin \frac{(l+1)\pi}{k+2}\right)^{2-2g} . \end{aligned} \tag{1.6}$$

As in [DW3], the strategy for proving Theorem 1.4 is to realize $\mathcal{V}^{g,k}_{n_1,\dots,n_m}$ as the space of holomorphic sections of a line bundle $\mathcal{A}^k(n_1, \dots, n_m)$ on $\mathcal{M}_{\alpha_1,\dots,\alpha_m}$, the moduli space of rank 2 bundles with parabolic structure and weight α_j at p_j, and then to study what happens when the surface degenerates. Let us briefly summarize the argument in the case of a non-separating cycle (the other case follows similarly): Suppose Σ_t is a surface of genus $g+1$ degenerating as $t \to 0$ to a surface of genus g with a double point. We think of the complex parameter t as measuring the pinching of a non-separating loop C in Σ_t. Now the theorem of Mehta and Seshadri [MS] identifies $\mathcal{M}_{\alpha_1,\dots,\alpha_m}$ with the equivalence classes of flat connections on $\Sigma_t \backslash \{p_1, \dots, p_m\}$ with holonomy conjugate to α_j around p_j, so let $r : \mathcal{M}_{\alpha_1,\dots,\alpha_m} \to [0, 1/2] \simeq \mathrm{SU}(2)/\mathrm{SU}(2)_{adj.}$ denote the holonomy map about C. Then for generic α, $r^{-1}(\alpha)$ is a smooth

hypersurface of real codimension 1, and there is an S^1 action making $r^{-1}(\alpha)$ into a principal circle bundle over $\mathcal{M}_{\alpha_1,\ldots,\alpha_m,\alpha,\alpha}$, the moduli of parabolic stable bundles on the normalization of the surface with a double point. Here, parabolic structure with weight α has been placed at the two points in the preimage of the node. Restricting sections of $\Delta^k(n_1,\ldots,n_m) \to \bar{\mathcal{M}}_{\alpha_1,\ldots,\alpha_m}$ to $r^{-1}(\alpha)$ and decomposing in terms of the weight l representations of the circle action gives sections of bundles $\Delta^k(n_1,\ldots,n_m,l,l) \to \mathcal{M}_{\alpha_1,\ldots,\alpha_m,\alpha,\alpha}$ and an isomorphism

$$\mathcal{V}^{g+1,k}_{n_1,\ldots,n_m} \simeq \bigoplus_{l=0}^{k} \mathcal{V}^{g,k}_{n_1,\ldots,n_m,l,l} \tag{1.7}$$

The main difficulties in proving (1.7) are: (1) showing that the restriction is an isomorphism between holomorphic sections on $\mathcal{M}_{\alpha_1,\ldots,\alpha_m}$ and CR-sections on the hypersurface $r^{-1}(\alpha)$; and (2) since the S^1 action is not holomorphic, except at $t = 0$, we must degenerate the surface, and it is therefore necessary to ensure that the dimension of the space of CR-sections does not jump. As discussed in [DW3], both of these issues depend on having eigenvalues of opposite signs for the Levi form of $r^{-1}(\alpha)$ (see Proposition 4.11 below), and indeed a similar factorization in the case of no punctures (i.e. $m = 0$) was the main result of [DW3]. In the present paper, we prove (1.7) for all m. This is the basic inductive step needed to prove Theorem 1.4.

Theorem 1.4 has independently been proven by Faltings [F] who uses a description of $\mathcal{V}^{g,k}_{n_1,\ldots,n_m}$ in terms of loop groups (see also [TUY], [KNR], [BL]). The result has also been obtained by Bertram [B1] who uses degeneration arguments in the context of stable pairs (see [BD] and [B2]) combined with the work of Szenes [Sz, BSz] who proves (1.6) and (1.7), and more generally the case of one puncture. The moduli space of stable pairs has been used by Thaddeus [T2] to prove (1.6) in a totally different manner. The proof of Theorem 1.4 presented in this note is independent of all the results cited here.

2 Vector bundles on punctured Riemann surfaces

In this section we discuss the different descriptions of the moduli spaces of vector bundles on punctured Riemann surfaces: as representations of the fundamental group, as L^2_δ connections, and as parabolic bundles. Most of what follows is a generalization of the case of one puncture described in [DW1-3], to which we refer for more details.

2.1 Representations of the fundamental group

Let Σ be a compact Riemann surface and $p_1,\ldots,p_m$ distinct points of Σ. Let $(\tilde{D}_i, z_i)$ be local coordinate systems around each p_i, i.e. $z_i : \tilde{D}_i \to D$ are complex analytic isomorphisms with the unit disk $D \subset \mathbb{C}$. We assume that $p_i = z_i^{-1}(0)$, and we denote by $\Sigma_{p_1,\ldots,p_m}$ the non-compact surface $\Sigma\backslash\{p_1,\ldots,p_m\}$.

We define new coordinates on $\tilde{D}_i^* = \tilde{D}_i \backslash \{p_i\}$ by $w_i = -\log z_i$. Then

$$w_i : \tilde{D}_i^* \to \mathbb{C}/(\tau, \theta) \sim (\tau, \theta + 2\pi)$$

maps $\tilde{D}_i^*$ analytically to the semi-infinite cylinder

$$C = \{(\tau, \theta) : \tau \geqq 0, 0 \leqq \theta \leqq 2\pi\}/(\tau, \theta) \sim (\tau, \theta + 2\pi) .$$

If we use polar coordinates (r, ϕ) on $D^* = D \backslash \{0\}$, then the map $w_i \circ z_i^{-1} : D^* \to \mathbb{C}$ is simply

$$\begin{cases} \tau = -\log r \\ \phi = \theta . \end{cases}$$

Let $P \to \Sigma_{p_1,\ldots,p_m}$ be the trivial SU(2) principal bundle and $E \to \Sigma_{p_1,\ldots,p_m}$ the hermitian vector bundle associated to the fundamental representation and the product metric. Let $\mathscr{A}$ be the space of hermitian connections on E and $\mathscr{A}_F$ the subspace of flat ones. Let $\mathscr{G} = \mathrm{Map}(\Sigma_{p_1,\ldots,p_m}, \mathrm{SU}(2))$ denote the real gauge group. We topologize both $\mathscr{A}$ and $\mathscr{G}$ with the obvious Fréchet topologies and set $\bar{\mathscr{R}} = \mathscr{A}_F/\mathscr{G}$.

There is an equivalent description of $\bar{\mathscr{R}}$ in terms of representations of the fundamental group of $\Sigma_{p_1,\ldots,p_m}$ as follows: First choose generators $\{\gamma_i\}_{i=1}^{2g}$ of $\pi_1(\Sigma)$ and simple closed curves $\lambda_1, \ldots, \lambda_m$ around $p_1, \ldots, p_m$ which do not intersect any of the γ_i's. We assume that the generators have been chosen such that they form a symplectic basis for $H_1(\Sigma)$. Then

$$\pi_1(\Sigma_{p_1,\ldots,p_m}) = \left\langle \gamma_i, \lambda_i : \prod_{i=1}^{g} [\gamma_i, \gamma_{i+g}] \lambda_1 \cdots \lambda_m = 1 \right\rangle .$$

The holonomy map identifies $\bar{\mathscr{R}}$ with the space of conjugacy classes of representations of $\pi_1(\Sigma_{p_1,\ldots,p_m})$ to SU(2).

Now choose weights $\alpha_1, \ldots, \alpha_m$ associated to the points $p_1, \ldots, p_m$ such that

$$0 < \alpha_i < 1/2 . \tag{2.1}$$

Let $\bar{\mathscr{R}}_{\alpha_1,\ldots,\alpha_m}$ denote the subspace of $\bar{\mathscr{R}}$ consisting of connections with holonomy $a_i = \exp(2\pi\sqrt{-1}A_i)$ around p_i, where

$$A_i = \begin{pmatrix} \alpha_i & 0 \\ 0 & -\alpha_i \end{pmatrix} . \tag{2.2}$$

In terms of representations,

$$\bar{\mathscr{R}}_{\alpha_1,\ldots,\alpha_m} = \{\rho \in \bar{\mathscr{R}} : \rho(\lambda_i) \text{ is conjugate to } a_i\} .$$

It follows that $\bar{\mathscr{R}}_{\alpha_1,\ldots,\alpha_m}$ is a real analytic variety of dimension $6g - 6 + 2m$. The varieties $\bar{\mathscr{R}}_{\alpha_1,\ldots,\alpha_m}$ are singular in general, but if we restrict to the irreducible representations we obtain a smooth open subvariety $\mathscr{R}_{\alpha_1,\ldots,\alpha_m}$ of $\bar{\mathscr{R}}_{\alpha_1,\ldots,\alpha_m}$.

2.2 The L^2_δ-construction

In this section we briefly describe a construction of the spaces $\mathscr{R}_{\alpha_1,\dots,\alpha_m}$ via weighted Sobolev spaces. In the case of one puncture our construction was carried out in [DW1, Sect. 3].

Let $\Sigma_{p_1,\dots,p_m}$ be as above. In addition, we make a choice of metric h on $\Sigma_{p_1,\dots,p_m}$ compatible with the complex structure and such that on $\tilde{D}_i^*$, $h = w_i^*(h_C)$, where $h_C = d\tau^2 + d\theta^2$ is the standard flat metric on the cylinder. Fix weights $\alpha_1,\dots,\alpha_m$ as in Sect. 2.1, and let A_i and a_i be the associated $\mathfrak{su}(2)$ and $SU(2)$ diagonal matrices (see (2.2)). Also let $\mathfrak{t}_{a_i} \subset \mathfrak{su}(2)$ denote the Lie algebra of the maximal torus $\mathbf{T}_{a_i} \subset SU(2)$ which contains a_i. Choose a connection ∇_0 on E such that on D_i^* it has the form $\nabla_0 = d + \sqrt{-1}A_i d\theta$. We will denote by ∇_{0,∂_i} the boundary operator of ∇_0 restricted to ∂D_i^*. We let

$$\mathscr{A}_\delta = \nabla_0 + L^2_{1,\delta}(T^*\Sigma_{p_1,\dots,p_m} \otimes \mathfrak{g}_E)$$

be the space of Sobolev $L^2_{1,\delta}$-connections. Following [DW1, Sect. 3] we define

$$Q = \{\varphi \in L^2_{2,\mathrm{loc.}}(\mathfrak{gl}(E)) : \|\nabla_0\varphi\|_{L^2_{1,\delta}} < \infty\},$$

and let

$$\mathscr{H} = \{\varphi \in Q : e^{-\tau\delta}\nabla_0^* e^{\tau\delta}\nabla_0\varphi = 0\}$$

be the subspace of harmonic ones. As in [DW1, Sect. 3] we have the splitting $Q = L^2_{2,\delta}(\mathfrak{gl}(E)) \oplus \mathscr{H}$. Moreover, there is a map $\sigma_i : Q \to \ker \nabla_{0,\partial_i} \simeq \mathfrak{t}_{a_i}$ given by

$$\sigma_i(\varphi)(\theta) = \lim_{\tau\to\infty} \varphi(w_i^{-1}(\tau,\theta)).$$

We set $\sigma = \prod_{i=1}^m \sigma_i$. Then $\sigma^{-1}(0) = L^2_{2,\delta}(\mathfrak{gl}(E))$ and

$$\sigma : \mathscr{H} \to \prod_{i=1}^m \ker \nabla_{0,\partial_i}$$

is an isomorphism.

Next we define the gauge groups

$$\mathscr{G}_\delta = \{\varphi \in Q : \varphi\varphi^* = \varphi^*\varphi = I, \det\varphi = I\}$$
$$\mathscr{G}_{0,\delta} = \{\varphi \in \mathscr{G}_\delta : \sigma(\varphi) = I\}.$$

It is not difficult to see that $\mathscr{G}_\delta$ and $\mathscr{G}_{0,\delta}$ are Banach Lie groups with Lie algebras

$$\mathrm{Lie}\,\mathscr{G}_\delta = \{\varphi \in Q : \varphi^* = -\varphi\}$$
$$\mathrm{Lie}\,\mathscr{G}_{0,\delta} = \{\varphi \in L^2_{1,\delta} : \varphi^* = -\varphi\}.$$

Moreover, there is an exact sequence of groups

$$1 \to \mathscr{G}_{0,\delta} \to \mathscr{G}_\delta \to \mathbf{T}_{a_1} \times \cdots \times \mathbf{T}_{a_m} \to 1. \tag{2.3}$$

Let $\mathscr{A}_{\delta,\mathrm{F}}$ denote the space of irreducible flat connections in $\mathscr{A}_\delta$. We set

$$\mathscr{F}^\delta_{\alpha_1,\dots,\alpha_m} = \mathscr{A}_{\delta,\mathrm{F}}/\mathscr{G}_{0,\delta}\,, \quad \mathscr{R}^\delta_{\alpha_1,\dots,\alpha_m} = \mathscr{A}_{\delta,\mathrm{F}}/\mathscr{G}_\delta$$

to be the quotient spaces. The product torus $\mathbf{T}_{a_1}\times\cdots\times\mathbf{T}_{a_m}$ acts on $\mathscr{F}^\delta_{\alpha_1,\dots,\alpha_m}$ via the sequence (2.3). More precisely, there is a splitting of (2.3) such that $(g_1,\dots,g_m)\in\mathbf{T}_{a_1}\times\cdots\times\mathbf{T}_{a_m}$ is mapped to the unique harmonic gauge transformation g with the property that for all $i=1,\dots,m$, $\lim_{\tau\to\infty} g(w_i^{-1}(\tau,\theta))=g_i$. From now on we will identify $\mathbf{T}_{a_1}\times\cdots\times\mathbf{T}_{a_m}$ with its image in $\mathscr{G}_\delta$ described above. Clearly,

$$\mathscr{R}^\delta_{\alpha_1,\dots,\alpha_m} = \mathscr{F}^\delta_{\alpha_1,\dots,\alpha_m}/\mathbf{T}_{a_1}\times\cdots\times\mathbf{T}_{a_m}\,.$$

The spaces $\mathscr{F}^\delta_{\alpha_1,\dots,\alpha_m}$ and $\mathscr{R}^\delta_{\alpha_1,\dots,\alpha_m}$ are naturally smooth manifolds, and we would like to briefly review their infinitesimal structure. Given $\nabla\in\mathscr{A}_{\delta,\mathrm{F}}$, let $\nabla^*=*\nabla*$ denote the L^2-adjoint and define the operator:

$$\begin{aligned}\delta_\nabla = (\nabla, e^{-\tau\delta}\nabla^* e^{\tau\delta}) : L^2_{1,\delta}(T^*\Sigma_{p_1,\dots,p_m}\otimes\mathfrak{g}_E)\\ \to L^2_\delta\left(\bigwedge^2 T^*\Sigma_{p_1,\dots,p_m}\otimes\mathfrak{g}_E\right)\oplus L^2_\delta(\mathfrak{g}_E)\,.\end{aligned}$$

Then for $\delta>0$ and sufficiently small, δ_∇ is a bounded Fredholm operator of index $6g-6+2m$ (cf. [DW1, Proposition 3.5]). Let $H^1_{\delta,\nabla}=\ker\delta_\nabla$, and let

$$\psi_\nabla : \mathscr{H}\to H^1_{\delta,\nabla} \tag{2.4}$$

be the map $\psi_\nabla(\varphi)=\nabla\varphi$. Let

$$H^1_{\delta^+,\nabla} = H^1_{\delta,\nabla}/\mathrm{image}(\psi_\nabla)\,.$$

The importance of these spaces is indicated in the following:

Proposition 2.5. (cf. [DW1, Theorem 3.7]) (i) $\mathscr{F}^\delta_{\alpha_1,\dots,\alpha_m}$ *is a smooth manifold of dimension* $6g-6+3m$. *Moreover, for* $[\nabla]\in\mathscr{F}^\delta_{\alpha_1,\dots,\alpha_m}$ *the tangent space is*

$$T_{[\nabla]}\mathscr{F}^\delta_{\alpha_1,\dots,\alpha_m}\simeq H^1_{\delta,\nabla}\,.$$

(ii) $\mathscr{R}^\delta_{\alpha_1,\dots,\alpha_m}$ *is a smooth manifold of dimension* $6g-6+2m$. *Moreover, for* $[\nabla]\in\mathscr{R}^\delta_{\alpha_1,\dots,\alpha_m}$ *the tangent space*

$$T_{[\nabla]}\mathscr{R}^\delta_{\alpha_1,\dots,\alpha_m}\simeq H^1_{\delta^+,\nabla}\,.$$

(iii) $\mathscr{R}^\delta_{\alpha_1,\dots,\alpha_m}$ *is diffeomorphic to* $\mathscr{R}_{\alpha_1,\dots,\alpha_m}$ (*henceforth, we will omit the* δ *and identify these spaces*).

In another direction, given $\nabla \in \mathscr{A}_{\delta,F}$ consider the unbounded operator

$$\bar{\partial}^*_{\nabla} : L^2(T^*\Sigma^{0,1}_{\alpha_1,\ldots,\alpha_m} \otimes \mathfrak{g}^{\mathbb{C}}_E) \to L^2(\mathfrak{g}^{\mathbb{C}}_E) .$$

It follows as in [DW1, Sect. 3] that $\ker \bar{\partial}^*_{\nabla}$ consists of (anti)-meromorphic $\mathfrak{g}^{\mathbb{C}}_E$-valued (0,1)-forms with at most simple poles and strictly upper triangular residues at $p_1,\ldots,p_m$. Moreover, as in [DW1, proof of Theorem 3.7],

$$\ker \bar{\partial}^*_{\nabla} \simeq H^1_{\delta^+,\nabla} , \tag{2.6}$$

and this defines an almost complex structure on $\mathscr{R}_{\alpha_1,\ldots,\alpha_m}$. In the next section we will see that this is indeed a complex structure, and furthermore it coincides with the complex structure defined by Mehta-Seshadri [MS]. Finally, we can use (2.6) to define a splitting

$$H^1_{\delta,\nabla} = \ker \bar{\partial}^*_{\nabla} \oplus \operatorname{image}(\psi_{\nabla}) \tag{2.7}$$

as in [DW1, proof of Theorem 3.7], where ψ_{∇} is the map (2.4). The splitting (2.7) will play an important role in the arguments of Sect. 4.

2.3 Parabolic bundles and the generalized Hecke correspondence

In order to show that the almost complex structure defined in Sect. 2.2 is a complex structure we need to introduce parabolic bundles and discuss the theorem of Mehta and Seshadri. For this, it will be convenient to consider holomorphic bundles $E \to \Sigma$ of arbitrary determinant $\bigwedge^2 E \simeq \Lambda$. When we wish to specify a holomorphic structure on the complex vector bundle E, i.e. a $\bar{\partial}$-operator $\bar{\partial}_E$, say, we shall write $E^{\bar{\partial}_E}$, although we will often omit this notation when the holomorphic structure is understood. Moreover, we shall think of a flag of E at p_i as a line ℓ_i in the fiber $E^*_{p_i}$. This is the dual of the flag defined in [DW1-3], but hopefully the change of notation (which now agrees with that of [B1-2]) will not cause confusion. Equivalently, a flag at p_i can be viewed as a map of sheaves

$$E \to \mathbb{C}_{p_i} \to 0 \tag{2.8}$$

where $\mathbb{C}_{p_i}$ denotes the torsion sheaf supported at p_i. Let $\ell_1,\ldots,\ell_m$ be flags at the points $p_1,\ldots,p_m$ and $\alpha_1,\ldots,\alpha_m$ weights as in (2.1).

Definition 2.9. *The triple* $(E,\{\ell_i\}_{i=1}^m,\{\alpha_i\}_{i=1}^m)$ *is called a parabolic bundle. It is called parabolic stable* (*resp. parabolic semi-stable*) *if for any holomorphic rank* 1 *subsheaf* $L \subset E$ *we have*

$$\deg L + \sum_{\ell_i(L)=0} \alpha_i - \sum_{\ell_j(L)\neq 0} \alpha_j \; < \; \tfrac{1}{2}\deg E \quad (\textit{resp.} \leqq) .$$

Observe that this definition is equivalent to the standard one (cf. [MS], [B1-2]). Also, note that the case $m = 0$, i.e. no punctures, reduces to the ordinary definition of stable and semi-stable bundles.

Given any *rational* weights $\alpha_1, \ldots, \alpha_m$ as in (2.1) one can apply the machinery of geometric invariant theory to construct a moduli space $\bar{\mathcal{M}}_{\alpha_1,\ldots,\alpha_m}(\Lambda)$ of parabolic semi-stable bundles with fixed determinant Λ which has the structure of a projective variety. The space $\bar{\mathcal{M}}_{\alpha_1,\ldots,\alpha_m}$ contains the open subvariety $\mathcal{M}_{\alpha_1,\ldots,\alpha_m}$ of parabolic stable bundles. The gauge theoretic construction of $\mathcal{M}_{\alpha_1,\ldots,\alpha_m}$ goes as follows. Fix a complex vector bundle $E \to \Sigma$ with $\bigwedge^2 E \simeq \Lambda$ (topologically), and fix flags $\ell_1, \ldots, \ell_m$ over the points $p_1, \ldots, p_m$. Let $\mathcal{A}_{p.s.}(\Lambda)$ denote the space of $\bar{\partial}$-operators on E which induce the fixed operator on Λ and are parabolic stable with respect to the flags $\ell_1, \ldots, \ell_m$. Let $\mathcal{G}^{\mathbb{C}}_{par.}$ denote the subgroup of the complex automorphism group of E consisting of the gauge transformations which induce the identity on Λ and preserve the flags $\ell_1, \ldots, \ell_m$. Then

$$\mathcal{M}_{\alpha_1,\ldots,\alpha_m}(\Lambda) \simeq \mathcal{A}_{p.s.}(\Lambda)/\mathcal{G}^{\mathbb{C}}_{par.} .$$

The next result is due to Mehta and Seshadri (cf. [MS], [DW1, Theorem 3.13])

Theorem 2.10. *For $\alpha_1, \ldots, \alpha_m \in \mathbb{Q} \cap (0, 1/2)$, $\mathcal{M}_{\alpha_1,\ldots,\alpha_m}(\Lambda)$ is a non-singular quasi-projective variety. Moreover, $\mathcal{M}_{\alpha_1,\ldots,\alpha_m}(\mathcal{O})$, where $\mathcal{O}$ is the trivial bundle, is diffeomorphic to $\mathcal{R}_{\alpha_1,\ldots,\alpha_m}$, and the complex structure agrees with the almost complex structure defined in Sect.* 2.2.

Special choices of the weights $\alpha_1, \ldots, \alpha_m$ will be important in this paper. We introduce some terminology:

Definition 2.11. *Let $\alpha_1, \ldots, \alpha_m$ be weights as in* (2.1). *Then we call $\alpha_1, \ldots, \alpha_m$ small if $\sum_{i=1}^m \alpha_i < 1/2$, and we will call them generic if $2\sum_{i=1}^m \varepsilon_i \alpha_i \notin \mathbb{Z}$, for every choice of $\varepsilon_1, \ldots, \varepsilon_m$, $\varepsilon_i = \pm 1$.*

The meaning of the above definition lies in the following rather obvious

Lemma 2.12. *Assume $\alpha_1, \ldots, \alpha_m$ are small and generic. Then parabolic semi-stability is equivalent to parabolic stability. Moreover, if we fix flags $\ell_1, \ldots, \ell_m$ of E. Then*

$$E \text{ stable} \Rightarrow (E, \{\ell_i\}, \{\alpha_i\}) \text{ parabolic stable} \Rightarrow E \text{ semi-stable} .$$

Finally, if $(E, \{\ell_i\}, \{\alpha_i\})$ is parabolic stable and $m \geqq 2$, then for any $j = 1, \ldots, m$, the triple

$$(E, \{\ell_1, \ldots, \hat{\ell}_j, \ldots, \ell_m\}, \{\alpha_1, \ldots, \hat{\alpha}_j, \ldots, \alpha_m\}) ,$$

where the circumflex indicates deletion, is also parabolic stable.

Corollary 2.13. *If $\alpha_1, \ldots, \alpha_m$ are generic then $\mathcal{M}_{\alpha_1,\ldots,\alpha_m}(\Lambda)$ is a smooth projective variety. Moreover, if $\alpha_1, \ldots, \alpha_m$ are small and generic and $m \geqq 2$,*

then for any $j = 1, \ldots, m$, there are "puncture forgetting" morphisms

$$\pi_j : \mathcal{M}_{\alpha_1,\ldots,\alpha_m}(\Lambda) \to \mathcal{M}_{\alpha_1,\ldots,\hat{\alpha}_j,\ldots,\alpha_m}(\Lambda) \tag{2.14}$$

whose fiber over a point $(E, \ell_1, \ldots, \hat{\ell}_j, \ldots, \ell_m)$ can be identified with $\mathbb{P}(E^*_{p_j})$. The maps π_j induce a morphism

$$\pi : \mathcal{M}_{\alpha_1,\ldots,\alpha_m}(\Lambda) \to \bar{\mathcal{M}}(\Lambda) \tag{2.15}$$

where $\bar{\mathcal{M}}(\Lambda)$ denotes the moduli of semi-stable bundles of determinant Λ. Moreover, if $E \in \bar{\mathcal{M}}(\Lambda)$ is stable then the fiber of π at E can be identified with $\mathbb{P}(E^*_{p_1}) \times \cdots \times \mathbb{P}(E^*_{p_m})$.

For rational choices of weights, Bertram proves in [B1, Corollary 3.4] that all the $\mathcal{M}_{\alpha_1,\ldots,\alpha_m}(\Lambda)$'s are mutually birational off a set of complex codimension 2 (assuming as usual that $g \geqq 2$). Therefore, we obtain

Corollary 2.16. *Let $\alpha_1, \ldots, \alpha_m$ be rational weights as in* (2.1). *Then there are rational maps*

$$\pi_j : \mathcal{M}_{\alpha_1,\ldots,\alpha_m}(\Lambda) \dashrightarrow \mathcal{M}_{\alpha_1,\ldots,\hat{\alpha}_j,\ldots,\alpha_m}(\Lambda) \tag{2.14'}$$

defined off a set of complex codimension 2 and whose fiber over a generic $(E, \ell_1, \ldots, \hat{\ell}_j, \ldots, \ell_m)$ can be identified with $\mathbb{P}(E^*_{p_j})$. The maps π_j induce a rational map

$$\pi : \mathcal{M}_{\alpha_1,\ldots,\alpha_m}(\Lambda) \dashrightarrow \bar{\mathcal{M}}(\Lambda) \tag{2.15'}$$

defined off a set of complex codimension 2 and whose generic fiber has the same description as the map (2.15).

We now specialize to the case $\Lambda \simeq \mathcal{O}$, the trivial bundle, and we set $\mathcal{M}_{\alpha_1,\ldots,\alpha_m} = \mathcal{M}_{\alpha_1,\ldots,\alpha_m}(\mathcal{O})$, $\mathcal{M} = \mathcal{M}(\mathcal{O})$. Complementary to the map (2.14′) are rational morphisms

$$\hat{\pi}_j : \mathcal{M}_{\alpha_1,\ldots,\alpha_m} \dashrightarrow \mathcal{M}_{\alpha_1,\ldots,\hat{\alpha}_j,\ldots,\alpha_m}(\mathcal{O}(-p_j)) \tag{2.17}$$

defined as follows. Assume for the moment that the weights are small and generic. Let $U \subset \mathcal{M}_{\alpha_1,\ldots,\alpha_m}$ be the subset $\pi^{-1}(\mathcal{M})$, where $\mathcal{M} \subset \bar{\mathcal{M}}$ is the subvariety of stable bundles. Let $(E, \{\ell_i\}, \{\alpha_i\}) \in U$. Recall that a flag ℓ_i is a map

$$\ell_i : E \to \mathbb{C}_{p_i} \to 0 .$$

Set $F = \ker \ell_j$. Then F is also a holomorphic rank 2 vector bundle, and we have the following

Lemma 2.18. *$(F, \ell_1, \ldots, \hat{\ell}_j, \ldots, \ell_m, \alpha_1, \ldots, \hat{\alpha}_j, \ldots, \alpha_m)$ is parabolic stable of determinant $\mathcal{O}(-p_j)$.*

Proof. Any subsheaf $L \subset F$ is also a subsheaf of E. Hence, by the stability of E, $\deg L \leqq -1$. It follows by the choice of weights α_i that

$$\deg L + \sum_{\substack{\ell_i(L')=0 \\ i \neq j}} \alpha_i - \sum_{\ell_i(L') \neq 0} \alpha_i < -1/2 ,$$

and therefore F is parabolic stable. The determinant follows from the definition of F.

□

We can thus define for $(E, \{\ell_i\}, \{\alpha_i\}) \in U$:

$$\hat{\pi}_j(E, \{\ell_i\}, \{\alpha_i\}) = (F, \ell_1, \ldots, \hat{\ell}_j, \ldots, \ell_m, \alpha_1, \ldots, \hat{\alpha}_j, \ldots, \alpha_m) .$$

Again, since all the $\mathcal{M}_{\alpha_1,\ldots,\alpha_m}$ are birational we have

Definition 2.19. *Let $\alpha_1, \ldots, \alpha_m$ be rational weights as in* (2.1). *We set*

$$\hat{\pi}_j : \mathcal{M}_{\alpha_1,\ldots,\alpha_m} \dashrightarrow \mathcal{M}_{\alpha_1,\ldots,\hat{\alpha}_j,\ldots,\alpha_m}(\mathcal{O}(-p_j)) \tag{2.20}$$

to be the maximal rational extension of the map $\hat{\pi}_j$ defined on $\pi^{-1}(U)$. The rational correspondence

$$\begin{array}{ccc} & \bar{\mathcal{M}}_{\alpha_1,\ldots,\alpha_m} & \\ {\scriptstyle \pi_j} \swarrow & & \searrow {\scriptstyle \hat{\pi}_j} \\ \mathcal{M}_{\alpha_1,\ldots,\hat{\alpha}_j,\ldots,\alpha_m} & & \mathcal{M}_{\alpha_1,\ldots,\hat{\alpha}_j,\ldots,\alpha_m}(\mathcal{O}(-p_j)) \end{array}$$

is called the generalized Hecke correspondence.

Remark 2.21. In the case $m = 1$, the maps $\pi_1, \hat{\pi}_1$ are defined on the whole of $\mathcal{M}_{\alpha_1}$ and the correspondence

$$\begin{array}{ccc} & \mathcal{M}_{\alpha_1} & \\ {\scriptstyle \pi_1} \swarrow & & \searrow {\scriptstyle \hat{\pi}_1} \\ \bar{\mathcal{M}} & & \mathcal{M}(\mathcal{O}(-p_1)) \end{array}$$

is the ordinary Hecke correspondence of Narasimhan and Ramanan (cf. [NR], [BSz], [DW2-3]).

2.4 Universal bundles and universal connections

An important feature of the spaces $\mathcal{M}_{\alpha_1,\ldots,\alpha_m}$ is that, unlike the case of $\mathcal{M}$, they always admit universal bundles

$$U^{(m)} \to \Sigma \times \mathcal{M}_{\alpha_1,\ldots,\alpha_m} . \tag{2.22}$$

These are constructed as follows. Let $\tilde{U}$ denote the pullback of $E \to \Sigma$ to $\Sigma \times \mathcal{A}_{p.s.}$ via projection onto the first factor. Then $\tilde{U}$ is endowed with a tautological holomorphic structure such that $\tilde{U}|_{\Sigma \times \{\bar{\partial}_E\}}$ is isomorphic to $E^{\bar{\partial}_E}$ as a bundle on Σ. Let $\tilde{L}_1 \to \Sigma \times \mathcal{A}_{p.s.}$ denote the trivial line bundle with the trivial holomorphic structure. Let $\mathcal{G}^{\mathbb{C}}_{par.}$ act on $\tilde{U}$ in the usual way and on $\tilde{L}_1$

via evaluation at p_1 and the fundamental representation. Now constant central elements of $\mathcal{G}^{\mathbb{C}}_{par.}$ act trivially on $\tilde{U}\otimes\tilde{L}_1$, so this bundle descends to a universal rank 2 bundle $U^{(m)} \to \Sigma \times \mathcal{M}_{\alpha_1,\dots,\alpha_m}$.

For future reference we also introduce the universal connection on the fibrations

$$\mathbf{T}_{a_1} \times \cdots \times \mathbf{T}_{a_m}/\mathbb{Z}_2 \to \mathcal{F}^{\delta}_{\alpha_1,\dots,\alpha_m} \to \mathcal{M}_{\alpha_1,\dots,\alpha_m} \simeq \mathcal{R}^{\delta}_{\alpha_1,\dots,\alpha_m} \tag{2.23}$$

obtained via the exact sequence of groups (2.3). Recall from Sect. 2.2 that $T_{[\nabla]}\mathcal{F}^{\delta}_{\alpha_1,\dots,\alpha_m} = H^1_{\delta,\nabla}$ and $T_{[\nabla]}\mathcal{M}_{\alpha_1,\dots,\alpha_m} = \ker \bar{\partial}^*_{\nabla}$. On the other hand we have seen that $\ker \bar{\partial}^*_{\nabla}$ consists of meromorphic (0,1)-forms in $\Sigma_{p_1,\dots,p_m}$ with values in $\mathfrak{g}^{\mathbb{C}}_E$ and having simple poles at $p_1,\dots,p_m$ with strictly upper triangular residues. It was shown in [DW1, Lemma 3.9] that such forms are in L^2_δ for δ sufficiently small. Consider the harmonic projection operator $\pi_\nabla : L^2_\delta \to H^1_{\delta,\nabla}$, and set

$$S^{\nabla}_{\alpha_1,\dots,\alpha_m} = \pi_\nabla(\ker \bar{\partial}^*_{\nabla}) \subset H^1_{\delta,\nabla}\,.$$

Clearly, π_∇ commutes with the action of the harmonic gauge transformations, hence $S^{\nabla}_{\alpha_1,\dots,\alpha_m}$ is invariant under the action of $\mathbf{T}_{a_1} \times \cdots \times \mathbf{T}_{a_m}$. It follows that the family of subspaces $S^{\nabla}_{\alpha_1,\dots,\alpha_m}$ defines a connection on the principal bundle (2.23). We will denote this connection by $S_{\alpha_1,\dots,\alpha_m}$ and call it the *universal connection*. As was shown in [DW3, Sect. 2.3] for the case of one puncture, we shall see in the proof of Proposition 4.11 that $S_{\alpha_1,\dots,\alpha_m}$ is in some sense the universal connection of Atiyah and Singer associated to the universal bundle $U^{(m)}$ (cf. [AS]).

3 Line bundles on moduli spaces and the vanishing theorem

We now define the line bundles on the space $\mathcal{M}_{\alpha_1,\dots,\alpha_m}$ which give rise to the multiplicity spaces mentioned in the introduction. These bundles depend on an integer k and choices of integers $n_1,\dots,n_m$ (which we think of as twice the spins of irreducible SU(2) representations) associated to each puncture. In Theorem 3.8 we generalize the main result of [DW2] to the effect that if any $n_i > k$ the line bundle has no non-trivial holomorphic sections. Instead of the topological arguments given in [DW2], we shall prove this result more directly.

Recall the definition (1.3) of the multiplicity spaces $\mathcal{V}^{g,k}_{n_1,\dots,n_m}$. These can be realized as spaces of holomorphic sections of line bundles on $\mathcal{M}_{\alpha_1,\dots,\alpha_m} = \mathcal{A}_{p.s.}/\mathcal{G}^{\mathbb{C}}_{par.}$ as follows: The integers $n_1,\dots,n_m$ define a character

$$\chi_{n_1,\dots,n_m} : \mathcal{G}^{\mathbb{C}}_{par.} \to \mathbb{C}^* \tag{3.1}$$

via evaluation at the points. More precisely, given $g \in \mathcal{G}^{\mathbb{C}}_{par.}$ the induced map $g(p_i)^* : E^*_{p_i} \to E^*_{p_i}$ defines (via an identification of $E^*_{p_i} \simeq \mathbb{C}^2$ and of the dual flag to ℓ_i with the standard flag in $\mathbb{C}^2$) an element of the Borel subgroup

$\mathbf{B}^+ \subset \mathrm{SL}(2,\mathbb{C})$. We set

$$\chi_{n_1,\dots,n_m}(g) = \chi_{n_1}(g(p_1)^*)\cdots\chi_{n_m}(g(p_m)^*)\,,$$

where $\chi_{n_i} : \mathbf{B}^+ \to \mathbb{C}^*$ is the character determined by the integer n_i.

Letting $\mathscr{G}^{\mathbb{C}}_{par.}$ act on the trivial line bundle $\mathscr{A}_{p.s.} \times \mathbb{C}$ via this character defines a line bundle on the quotient $\mathscr{M}_{\alpha_1,\dots,\alpha_m}$ provided that the character is invariant under the isotropy group, i.e. provided that the sum of the n_i's is even. We call this line bundle $L(n_1,\dots,n_m)$. Recall also the map π of (2.15′). Then $\pi^*\Delta$, where Δ is the determinant line bundle on $\mathscr{M}(\mathscr{O})$, extends uniquely to a line bundle on $\mathscr{M}_{\alpha_1,\dots,\alpha_m}$. Set

$$\Delta^k(n_1,\dots,n_m) = \pi^*\Delta^k \otimes L(n_1,\dots,n_m)\,.$$

Then we have

Proposition 3.2. *There is an isomorphism*

$$\mathscr{V}^{g,k}_{n_1,\dots,n_m} \simeq H^0(\mathscr{M}_{\alpha_1,\dots,\alpha_m}, \Delta^k(n_1,\dots,n_m))\,,$$

where $\mathscr{V}^{g,k}_{n_1,\dots,n_m}$ *is defined in* (1.3).

Proof. As mentioned above, the $\mathscr{M}_{\alpha_1,\dots,\alpha_m}$ are all mutually birational off a set of codimension at least 2. Therefore it suffices to prove the result for small weights. Let

$$\mathscr{B} \subset \mathscr{A} \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$$

be the set of points $(\bar\partial_E, \ell_1,\dots,\ell_m)$ such that $E^{\bar\partial_E}$ is parabolic stable with respect to the weights $\alpha_1,\dots,\alpha_m$ and the flags $\ell_1,\dots,\ell_m$. Notice that $\mathscr{B}/\mathscr{G}^{\mathbb{C}} \simeq \mathscr{A}_{p.s.}/\mathscr{G}^{\mathbb{C}}_{par.}$. The space $\mathscr{V}^{g,k}_{n_1,\dots,n_m}$ may be identified with $\mathscr{G}^{\mathbb{C}}$-invariant holomorphic sections of the line bundle $\Delta^k \otimes B_{n_1} \otimes \cdots \otimes B_{n_m}$ on $\mathscr{A} \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ obtained from the determinant bundle on $\mathscr{A}$ and the Borel-Weil-Bott bundles on $\mathbb{P}^1$ (cf. [DW1]). We next claim that the complement of $\mathscr{B}$ in $\mathscr{A} \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ has codimension at least 2. Since the weights are assumed to be small,

$$\mathscr{A}_s \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \subset \mathscr{B} \subset \mathscr{A}_{ss} \times \mathbb{P}^1 \times \cdots \times \mathbb{P}^1\,,$$

where $\mathscr{A}_s$ and $\mathscr{A}_{ss}$ denote the stable and semi-stable points of $\mathscr{A}$, respectively (see Lemma 2.12). For $g \geqq 2$ the complement of $\mathscr{A}_{ss}$ in $\mathscr{A}$ has codimension at least 2 (see [AB]). On the other hand, the strictly semi-stable points $\mathscr{A}_{ss} \setminus \mathscr{A}_s$ have codimension at least 1 in $\mathscr{A}_{ss}$ (cf. [S]), and if a bundle E is strictly semi-stable, then it is easy to see that it is parabolic stable for a Zariski dense choice of flags. The claim then follows. Thus we have

$$\begin{aligned}\mathscr{V}^{g,k}_{n_1,\dots,n_m} &\simeq H^0(\mathscr{B}, \Delta^k \otimes B_{n_1} \otimes \cdots \otimes B_{n_m})^{\mathscr{G}^{\mathbb{C}}} \\ &= H^0(\mathscr{A}_{p.s.}, \Delta^k(n_1,\dots,n_m))^{\mathscr{G}^{\mathbb{C}}_{par.}} \\ &= H^0(\mathscr{M}_{\alpha_1,\dots,\alpha_m}, \Delta^k(n_1,\dots,n_m))\,,\end{aligned}$$

and this proves the proposition. □

Lemma 3.3. *The restriction of* $L(n_1,\dots,n_m)$ *to the generic fiber of the map* $\hat{\pi}_j$ *of* (2.20) *is* $\mathcal{O}_{\mathbb{P}^1}(-n_j)$.

Proof. Fix $(F,\ell_1,\dots,\hat{\ell}_j,\dots,\ell_m)$ a generic point in $\mathcal{M}_{\alpha_1,\dots,\hat{\alpha}_j,\dots,\alpha_m}(\mathcal{O}(-p_j))$. The fiber of $\hat{\pi}_j$ over this point can be identified with $\mathbb{P}(\mathrm{Ext}^1(\mathbb{C}_{p_j},F))$. Let $\hat{\imath}_j$ denote the natural inclusion of the fiber. Then the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Ext}^1(\mathbb{C}_{p_j},F)\backslash\{0\} & \xrightarrow{\tilde{\imath}_j} & \mathcal{A}_{p.s.} \\ \downarrow\tilde{p} & & \downarrow\hat{p} \\ \mathbb{P}(\mathrm{Ext}^1(\mathbb{C}_{p_j},F)) & \xrightarrow{\hat{\imath}_j} & \bar{\mathcal{M}}_{\alpha_1,\dots,\alpha_m}=\mathcal{A}_{p.s.}/\mathcal{G}^{\mathbb{C}}_{par.} \end{array} \tag{3.4}$$

where $\hat{p}$ and $\tilde{p}$ are the obvious projection maps and $\tilde{\imath}_j$ the natural inclusion. Observe that for $\lambda\in\mathbb{C}^*$ and $\xi\in\mathrm{Ext}^1(\mathbb{C}_{p_j},F)\backslash\{0\}$,

$$\tilde{\imath}_j(\lambda\xi)=g_\lambda\tilde{\imath}_j(\xi)\,, \tag{3.5}$$

where $g_\lambda=\mathrm{diag}(\lambda,\lambda^{-1})\in\mathcal{G}^{\mathbb{C}}_{par.}$. This can be verified as follows. Given an extension ξ corresponding to the bundle E, we have an exact sequence of sheaves

$$0\to F\to E\to\mathbb{C}_{p_j}\to 0\,. \tag{3.6}$$

By restricting to the fiber at p_j we obtain a four term exact sequence

$$0\to\mathbb{C}\xrightarrow{f}F_{p_j}\to E_{p_j}\xrightarrow{\sigma}\mathbb{C}\to 0\,,$$

where f can be identified with ξ via the isomorphism

$$\mathrm{Ext}^1(\mathbb{C}_{p_j},F)\simeq F_{p_j}\,.$$

Moreover, the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{C} & \xrightarrow{f} & F_{p_j} & \to & E_{p_j} & \xrightarrow{\sigma} & \mathbb{C} & \to & 0 \\ & & \| & & \downarrow g_\lambda & & \downarrow g_\lambda & & \| & & \\ 0 & \to & \mathbb{C} & \xrightarrow{\lambda\cdot f} & F_{p_j} & \to & E_{p_j} & \xrightarrow{\lambda\cdot\sigma} & \mathbb{C} & \to & 0\,, \end{array}$$

so (3.5) follows. Thus, the character $\chi_{n_1,\dots,n_m}$ restricts to the character λ^{n_j}, and hence $L(n_1,\dots,n_m)$ restricts to $\mathcal{O}_{\mathbb{P}^1}(-n_j)$ on $\mathbb{P}(\mathrm{Ext}^1(\mathbb{C}_{p_j},F))$. The lemma follows. □

Lemma 3.7. *Let* π *be the map* (2.15′). *Then the restriction of* $\pi^*\Delta$ *to the generic fiber of the map* $\hat{\pi}_j$ *of* (2.20) *is* $\mathcal{O}_{\mathbb{P}^1}(1)$.

Proof. By taking the long exact sequence in cohomology associated to (3.6) we obtain

$$\det\bar{\partial}_E\simeq\det\bar{\partial}_F\otimes H^0(\mathbb{C}_{p_j})^*$$

(recall that the definition of the determinant line bundle gives the *dual* of the determinant of the index bundle (cf. [D])). If we lift $\pi^*\Delta$ to $\mathcal{A}_{p.s.}$ and restrict to the extension ξ corresponding to E in $\mathrm{Ext}^1(\mathbb{C}_{p_j},F)$ we obtain $\det\bar{\partial}_E$. If we

rescale the extension ξ by $\lambda \in \mathbb{C}^*$ then the action on $\det \bar{\partial}_F$ is trivial, since F is fixed, whereas the action on $H^0(\mathbb{C}_{p_j}) \simeq \mathbb{C}$ is given by multiplication by λ. Hence the action on $\det \bar{\partial}_E$ is given by multiplication by λ^{-1}. The lemma follows. □

Taken together, Lemmas 3.3 and 3.7 imply the following generalization of the main theorem of [DW2].

Theorem 3.8. *For any choice of integers $n_1, \ldots, n_m$, not necessarily positive, let*

$$D^{g,k}_{n_1,\ldots,n_m} = \dim H^0(\mathcal{M}_{\alpha_1,\ldots,\alpha_m}, \Delta^k(n_1,\ldots,n_m)),$$

if $\sum n_i$ is even, and set $D^{g,k}_{n_1,\ldots,n_m} = 0$ otherwise. Then the numbers $D^{g,k}_{n_1,\ldots,n_m}$ are independent of the weights $\alpha_1,\ldots,\alpha_m$, the locations of the points $p_1,\ldots,p_m$, and the complex structure on Σ. If any $n_j < 0$ or $n_j > k$, then $D^{g,k}_{n_1,\ldots,n_m} = 0$. In particular, $\dim \mathcal{V}^{g,k}_{n_1,\ldots,n_m} = 0$ if any $n_j > k$.

Proof. For the vanishing, by Lemmas 3.3 and 3.7, the restriction of $\Delta^k(n_1,\ldots,n_m)$ to the fiber of $\hat{\pi}_j$ is $\mathcal{O}_{\mathbb{P}^1}(k - n_j)$. Since $\hat{\pi}_j$ is defined on an open dense set, there are no non-trivial holomorphic sections for $n_j > k$. A similar argument for $n_j < 0$ applies using the map π_j of (2.14). The independence of parameters follows from the fact that the $\mathcal{M}_{\alpha_1,\ldots,\alpha_m}$'s are mutually birational off a set of codimension at least 2, and for $0 \leqq n_1,\ldots,n_m \leqq k$ and a particular choice of weights $\alpha_1,\ldots,\alpha_m$ (depending on $n_1,\ldots,n_m$) one can show that

$$H^1(\mathcal{M}_{\alpha_1,\ldots,\alpha_m}, \Delta^k(n_1,\ldots,n_m)) = 0,$$

(see [B1, Corollary 3.5]). □

We end this section by discussing some results which will be used later. Let

$$U^{(m)*} \to \Sigma \times \mathcal{M}_{\alpha_1,\ldots,\alpha_m}$$

be the dual of the universal bundle constructed in Sect. 2.4, and let $P^{(m)}$ be the associated SO(3)-bundle. Let

$$P^{(m)}_{p_{m+1}} = P^{(m)}|_{\{p_{m+1}\}\times \mathcal{M}_{\alpha_1,\ldots,\alpha_m}}.$$

Then for a choice of maximal torus $\mathbf{T}_a \subset \mathrm{SU}(2)$

$$\mathbf{T}_a/\mathbb{Z}_2 \to P^{(m)}_{p_{m+1}} \to P^{(m)}_{p_{m+1}}/\mathbf{T}_a \tag{3.9}$$

defines a principal $S^1 \simeq \mathbf{T}_a/\mathbb{Z}_2$ bundle, and $P^{(m)}_{p_{m+1}}/\mathbf{T}_a$ may be identified with $\mathcal{M}_{\alpha_1,\ldots,\alpha_m,\alpha}$ for α sufficiently small. We claim

Lemma 3.10. *The line bundle associated to* (3.9) *via the standard representation of $\mathbf{T}_a/\mathbb{Z}_2$ is isomorphic to $L(0,\ldots,0,2)$.*

Proof. Let $\mathscr{G}^{\mathbb{C}}_{par.,I} = \{g \in \mathscr{G}^{\mathbb{C}}_{par.} : g(p_{m+1}) = I\}$. It follows that

$$\mathscr{A}_{p.s.}/\mathscr{G}^{\mathbb{C}}_{par.,I} \to \mathscr{A}_{p.s.}/\mathscr{G}^{\mathbb{C}}_{par.} = \mathscr{M}_{\alpha_1,\dots,\alpha_m,\alpha} \tag{3.11}$$

is a principal $\mathbf{B}^+/Z(\mathbf{B}^+)$-bundle, where $\mathbf{B}^+ \subset \mathrm{SL}(2,\mathbb{C})$ denotes the Borel subgroup and Z its center. It also follows that the fibration (3.9) is a $\mathbf{T}_a/\mathbb{Z}_2 \subset \mathbf{B}^+/Z(\mathbf{B}^+)$ reduction of (3.11). Moreover, the character $\chi_{0,\dots,0,2}$ defined in (3.1) corresponds to the standard character $\chi : \mathbf{T}_a/\mathbb{Z}_2 \to \mathbb{C}^*$. The lemma follows. □

Next recall the principal circle bundle

$$\mathbf{T}_a/\mathbb{Z}_2 \to \mathscr{F}^{\delta}_{\alpha_1,\dots,\alpha_m,\alpha}/\mathbf{T}_{a_1} \times \cdots \times \mathbf{T}_{a_m} \to \mathscr{M}_{\alpha_1,\dots,\alpha_m,\alpha} \tag{3.12}$$

Proposition 3.13. *The principal S^1-bundles* (3.9) *and* (3.12) *are isomorphic. In particular, the line bundle associated to* (3.12) *via the standard representation of $\mathbf{T}_a/\mathbb{Z}_2$ is $L(0,\dots,0,2)$.*

Proof. An explicit $\mathbf{T}_a/\mathbb{Z}_2$-equivariant isomorphism between $\mathscr{F}^{\delta}_{\alpha_1,\dots,\alpha_m,\alpha}/\mathbf{T}_{a_1} \times \cdots \times \mathbf{T}_{a_m}$ and $P^{(m)}_{p_{m+1}}$ can be constructed as follows: Given $[\nabla] \in \mathscr{F}^{\delta}_{\alpha_1,\dots,\alpha_m,\alpha}/\mathbf{T}_{a_1} \times \cdots \times \mathbf{T}_{a_m}$, choose a representative $\nabla \in \mathscr{A}_{\delta,\mathrm{F}}$ such that $\nabla = d + i\hat{\alpha}_j d\theta$ around p_j, $j = 1,\dots,m+1$, and where we set $\hat{\alpha}_{m+1} = \hat{\alpha}$. Let $\bar{\partial}_\nabla$ denote the associated $\bar{\partial}$-operator, and let g be a (singular) complex gauge transformation such that

$$g(z) = \begin{cases} I & \text{if } z \in \Sigma_{p_1,\dots,p_m,p_{m+1}} \setminus \cup_{i=1}^{m+1} \tilde{D}_i^* \,, \\ \mathrm{diag}(|z|^{\alpha_j/2}, |z|^{-\alpha_j/2}) & \text{if } z \text{ is near } p_j \,. \end{cases}$$

By the Mehta-Seshadri theorem, $g(\bar{\partial}_\nabla) \in \mathscr{A}_{p.s.}$; let $[g(\bar{\partial}_\nabla)]$ denote the class of $g(\bar{\partial}_\nabla)$ in $\mathscr{A}_{p.s.}/\mathscr{G}^{\mathbb{C}}_{par.,I} = P^{(m)}_{p_{m+1}}$. Then the map $[\nabla] \mapsto [g(\bar{\partial}_\nabla)]$ defines the desired $\mathbf{T}_a/\mathbb{Z}_2$-equivariant isomorphism. □

Given spaces $\mathscr{F}^{+,\delta}_{\alpha_1,\dots,\alpha_{m^+},\alpha}/\mathbf{T}_{a_1} \times \cdots \times \mathbf{T}_{a_{m^+}}$ and $\mathscr{F}^{-,\delta}_{\alpha_{m^+1},\dots,\alpha_m,\alpha}/\mathbf{T}_{a_{m^+1}} \times \cdots \times \mathbf{T}_{a_m}$ associated to Riemann surfaces Σ^+ and Σ^- with corresponding marked points, we can form the principal bundle

$$\begin{array}{c} S^1 \simeq \mathbf{T}_a/\mathbb{Z}_2 \\ \downarrow \\ \{\mathscr{F}^{+,\delta}_{\alpha_1,\dots,\alpha_{m+},\alpha}/\mathbf{T}_{a_1} \times \cdots \times \mathbf{T}_{a_{m^+}}\} \times_{\mathbf{T}_a} \{\mathscr{F}^{-,\delta}_{\alpha_{m^+1},\dots,\alpha_m,\alpha}/\mathbf{T}_{a_{m^+1}} \times \cdots \times \mathbf{T}_{a_m}\} \\ \downarrow \\ \mathscr{M}^{+}_{\alpha_1,\dots,\alpha_m,\alpha} \times \mathscr{M}^{-}_{\alpha_1,\dots,\alpha_m,\alpha} \end{array} \tag{3.14}$$

where $\mathbf{T}_a$ acts diagonally. Then we have

Proposition 3.15. *The line bundle associated to* (3.14) *via the standard representation of $\mathbf{T}_a/\mathbb{Z}_2$ is isomorphic to the exterior tensor product $L^+(0,\dots,0,2) \otimes L^-(0,\dots,0,2)$.*

Similarly, let

$$P = P^{(m)}_{p_{m+1}} \times_{SO(3)} P^{(m)}_{p_{m+2}} . \tag{3.16}$$

Then P defines a principal $S^1 \simeq \mathbf{T}_a/\mathbf{Z}_2$-bundle

$$\mathbf{T}_a/\mathbf{Z}_2 \to P \to P/\mathbf{T}_a , \tag{3.17}$$

Clearly, $P/\mathbf{T}_a$ can be identified with $\mathscr{M}_{\alpha_1,\dots,\alpha_m,\alpha,\alpha}$, and as in Lemma 3.10 we have

Lemma 3.18. *The induced line bundle from* (3.17) *via the standard representation of* $\mathbf{T}_a/\mathbf{Z}_2$ *is isomorphic to* $L(0,\dots,0,1,1)$.

Finally, recall the principal $S^1 \simeq \mathbf{T}_a/\mathbf{Z}_2$-bundle

$$\mathbf{T}_a/\mathbf{Z}_2 \to \mathscr{F}^{\delta}_{\alpha_1,\dots,\alpha_m,\alpha,\alpha}/\mathbf{T}_{a_1} \times \cdots \times \mathbf{T}_{a_m} \times \mathbf{T}_a \to \mathscr{M}_{\alpha_1,\dots,\alpha_m,\alpha,\alpha} , \tag{3.19}$$

where the last $\mathbf{T}_a$ acts diagonally. The analogue of Proposition 3.15 is then

Proposition 3.20. *The principal* S^1*-bundles* (3.17) *and* (3.19) *are isomorphic. In particular, the line bundle associated to* (3.19) *via the standard representation of* $\mathbf{T}_a/\mathbf{Z}_2$ *is* $L(0,\dots,0,1,1)$.

4 Semicontinuity of Verlinde dimensions

The purpose of this section is to prove the following inequalities:

Theorem 4.1. *For* $g \geqq 2$ *and any choice of integers* $n_1,\dots,n_m$, *then*

$$D^{g+1,k}_{n_1,\dots,n_m} \leqq \sum_{l=0}^{k} D^{g,k}_{n_1,\dots,n_m,l,l} .$$

Theorem 4.2. *If* $g_+ \geqq 2$, $g_- \geqq 1$, $g = g_+ + g_-$, *and* $1 \leqq m^+ \leqq m$, *then*

$$D^{g,k}_{n_1,\dots,n_m} \leqq \sum_{l=0}^{k} D^{g_+,k}_{n_1,\dots,n_{m^+},l} D^{g_-,k}_{n_{m^+}+1,\dots,n_m,l} .$$

Both Theorems 4.1 and 4.2 are proved by degeneration. We shall argue in Sect. 5 that *the inequalities in the theorems above are in fact equalities.* Our method is almost certainly strong enough to prove this result directly, but we have not gone to the trouble of establishing the vanishing theorems necessary to carry this out. Instead, we note that for the situation of Theorem 4.2 with no punctures ($m = 0$) and $g_\pm \geqq 2$, $D^{g,k}$ is equal to the decomposition on the right hand side. This was the main theorem of [DW3] (see also Theorem 4.27 below), and it turns out that this is all we need to prove that the general case also yields equalities (see Sect. 5). For reference, we also remark that the quantities $N^{g,k}_{n_1,\dots,n_m}$ defined in (1.2) satisfy the "factorization rules" above (cf. [V], [MoSb]),

$$N^{g+1,k}_{n_1,\dots,n_m} = \sum_{l=0}^{k} N^{g,k}_{n_1,\dots,n_m,l,l} , \tag{4.3}$$

$$N^{g,k}_{n_1,\dots,n_m} = \sum_{l=0}^{k} N^{g_+,k}_{n_1,\dots,n_{m^+},l} N^{g_-,k}_{n_{m^+}+1,\dots,n_m,l} . \tag{4.4}$$

4.1 *Proof of Theorem 4.1*

Let $\Sigma_{p_1,\dots,p_m}$ denote a compact Riemann surface of genus $g \geqq 2$ punctured at $p_1,\dots,p_m$. Let p_{m+1}, p_{m+2} be distinct points of $\Sigma_{p_1,\dots,p_m}$ and

$$z_i : \tilde{D}_i \to D \tag{4.5}$$

be disjoint coordinate systems around p_i, $p_i = z_i^{-1}(0)$ for $i = m+1, m+2$. Given a complex number t in the punctured disk D^* let $\Sigma_{p_1,\dots,p_m}(t)$ denote the Riemann surface of genus $g+1$ obtained by glueing in the usual way via the equation $z_{m+1}z_{m+2} = t$ (cf. [DW3]). Let $\varepsilon = |t|^{1/2}$ and let C_ε denote the circle in $\Sigma_{p_1,\dots,p_m}(t)$ defined by the equation $|z_{m+1}| = |z_{m+2}| = \varepsilon$. Then C_ε is a non-separating cycle of $\Sigma_{p_1,\dots,p_m}(t)$. We give C_ε the boundary orientation from $\Sigma_{p_1,\dots,p_m}\backslash\{|z_{m+1}| < \varepsilon\}$, which is opposite to that coming from $\Sigma_{p_1,\dots,p_m}\backslash\{|z_{m+2}| < \varepsilon\}$. Let $\Sigma_{p_1,\dots,p_{m+2}} = \Sigma_{p_1,\dots,p_m}\backslash\{p_{m+1}, p_{m+2}\}$. We also fix a family of diffeomorphisms

$$\chi_t : \Sigma_{p_1,\dots,p_{m+2}} \to \Sigma_{p_1,\dots,p_m}(t)\backslash C_\varepsilon$$

such that $\chi_0 = \mathrm{id}$ and $\chi_t = \mathrm{id}$ outside the pinching region $\{|t| < |z_{m+1}|, |z_{m+2}| < 1\}$.

Notation 4.6. For simplicity we shall use the following notation *for this subsection only*:

$$\Sigma_t = \Sigma_{p_1,\dots,p_m}(t)\,, \quad \Sigma_0 = \Sigma_{p_1,\dots,p_{m+2}}\,.$$

Also, given rational weights $\alpha_1,\dots,\alpha_m$ as in (2.1) we set

$$\mathcal{M}(t) = \mathcal{M}_{\alpha_1,\dots,\alpha_m}(t)$$

and

$$r : \mathcal{M}(t) \to [0, 1/2] \simeq \mathrm{SU}(2)/\mathrm{SU}(2)_{adj.}$$

for the holonomy map around C_ε.

The next result is straightforward.

Lemma 4.7. *The map r is continuous and it is smooth over the preimage of the open interval $(0, 1/2)$. Moreover, r has only finitely many critical values.*

The critical values can only occur for those α for which $\alpha_1,\dots,\alpha_m,\alpha,\dot{\alpha}$ are non-generic (see Definition 2.11).

Let α be a regular value of r (which we may take to be a rational number by Lemma 4.7). Then $\mathcal{M}_\alpha(t) = r^{-1}(\alpha)$ has the structure of a CR-manifold. Indeed, for $[\nabla_t] \in \mathcal{M}_\alpha(t)$ we set

$$S^\alpha_{[\nabla_t]} = T_{[\nabla_t]}\mathcal{M}_\alpha(t) \cap *_t T_{[\nabla_t]}\mathcal{M}_\alpha(t)\,, \tag{4.8}$$

where $*_t$ denotes the complex structure of $\mathcal{M}(t)$ which is induced by the $*_t$-operator on Σ_t. Then $S^\alpha \subset T\mathcal{M}_\alpha(t)$ defines a CR-structure on $\mathcal{M}_\alpha(t)$.

Let $\Delta^k(n_1,\dots,n_m)(t)$ denote the holomorphic line bundle defined in Sect. 3. For α as above, we set

$$L_\alpha(t) = \Delta^k(n_1,\dots,n_m)(t)|_{\mathscr{M}_\alpha(t)}\,.$$

Then $L_\alpha(t)$ is a CR-line bundle over $\mathscr{M}_\alpha(t)$, and since $\mathscr{M}_\alpha(t)$ has real codimension 1 in $\mathscr{M}(t)$ the restriction map

$$\rho : H^0(\mathscr{M}(t), \Delta^k(n_1,\dots,n_m)(t)) \to H^0_{\mathrm{CR}}(\mathscr{M}_\alpha(t), L_\alpha(t))$$

is injective. In particular,

$$(4.9)\quad D^{g+1,k}_{n_1,\dots,n_m} = \dim H^0(\mathscr{M}(t), \Delta^k(n_1,\dots,n_m)(t)) \leqq \dim H^0_{\mathrm{CR}}(\mathscr{M}_\alpha(t), L_\alpha(t))\,,$$

where à priori the right hand side may be infinity. In order to relate this to the right hand side of the formula appearing in Theorem 4.1, we will have to degenerate Σ_t to $t = 0$. Our arguments closely follow those in [DW3].

Recall from Sect. 2.2 that for $[\nabla_t] \in \mathscr{M}(t)$, $T_{[\nabla_t]}\mathscr{M}(t)$ can be identified with the subspace $K_{\nabla_t}(t)$ of $L^2_{-\delta}(T^*\Sigma_t^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}})$ consisting of meromorphic (0,1) forms on Σ_t having at most simple poles with strictly upper triangular residues at p_i. Moreover, by using the identification $\chi_t : \Sigma_0 \to \Sigma_t \backslash C_\epsilon$, we can view $K_{\nabla_t}(t)$ as subspaces of $L^2_{-\delta}(T^*\Sigma_0^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}})$ (cf. [DW3, (3.22)]). Furthermore, the degeneration theorem of [DW3, Sect. 3] is local around the punctures, so as in [DW3, Corollary 3.24, Proposition 4.13] we have that

$$K_{\nabla_t}(t) \to K_{\nabla_0}(0)$$

is a continuous family of vector spaces in $L^2_{-\delta}(T^*\Sigma_0^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}})$, where K_{∇_0} consists of meromorphic (0,1) forms β on Σ_0 with at most simple poles with residues strictly upper triangular at $p_1,\dots,p_m$, upper triangular at p_{m+1}, p_{m+2}, and satisfying the matching condition

$$\mathrm{proj}_{\mathfrak{t}_{a_{m+1}}} \mathrm{res}_{p_{m+1}}\beta = \mathrm{proj}_{\mathfrak{t}_{a_{m+2}}} \mathrm{res}_{p_{m+2}}\beta\,.$$

Now assume that $\nabla_t \in \mathscr{M}_\alpha(t)$. Then

$$T_{\nabla_t}\mathscr{M}_\alpha(t) \simeq K^\alpha_{\nabla_t}(t) = K_{\nabla_t}(t) \cap \ker d^r \subset L^2_{-\delta}(T^*\Sigma_0^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}})\,,$$

and as in [DW3, Corollary 4.17],

$$S^\alpha_{\nabla_t}(t) = K^\alpha_{\nabla_t}(t) \cap *_t \ker K^\alpha_{\nabla_t}(t) \to S^\alpha_{\nabla_0}(0)$$

is a continuous family of subspaces in $L^2_{-\delta}(T^*\Sigma_0^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}})$, where $S^\alpha_{\nabla_0}(0)$ consists of meromorphic (0,1) forms with at most simple poles and strictly upper triangular residues at $p_1,\dots,p_{m+2}$. However, $S^\alpha_{\nabla_0}(0) \simeq H^1_{\delta^+,\nabla_0}$ can be viewed as a subspace of

$$T_{\nabla_0}(\mathscr{F}^\delta_{\alpha_1,\dots,\alpha_m,\alpha,\alpha}/\mathbf{T}_{a_1} \times \cdots \times \mathbf{T}_{a_m} \times \mathbf{T}_a)\,,$$

where the last $\mathbf{T}_a$ acts diagonally. Therefore, if we define the limiting CR-structure on $\mathscr{F}^\delta_{\alpha_1,\dots,\alpha_m,\alpha,\alpha}/\mathbf{T}_{a_1} \times \cdots \times \mathbf{T}_{a_m} \times \mathbf{T}_a$ by $S^\alpha(0)$, then we obtain as in [DW3, Theorem 4.11]

Proposition 4.10. *Let* $\mathfrak{M}_\alpha = \cup_{t\in D}\mathscr{M}_\alpha(t)$. *Then* $\mathfrak{M}_\alpha$ *is a differentiable family of CR-manifolds parametrized by the disk.*

As a consequence of Proposition 4.10 we have

Proposition 4.11. *The Levi form of* $\mathscr{M}_\alpha(0)$ *has everywhere at least two positive and two negative eigenvalues.*

The existence of two eigenvalues of each sign was crucial for the proof of continuity in the degeneration of [DW3]. In the general case, however, the relevant H^1 group cannot be shown to vanish at $t = 0$, and it is for this reason that by direct arguments we are only able to obtain semicontinuity.

Proof. Recall from Sect. 2.4 that the fibration

$$\mathscr{F}^\delta_{\alpha_1,\ldots,\alpha_m,\alpha,\alpha} \to \mathscr{M}_{\alpha_1,\ldots,\alpha_m,\alpha,\alpha}$$

has a "universal" connection $S_{\alpha_1,\ldots,\alpha_m,\alpha,\alpha}$. By dividing out by $\mathbf{T}_{a_1}\times\cdots\times\mathbf{T}_{a_m}\times\mathbf{T}_a$ where the last $\mathbf{T}_a$ acts diagonally we obtain a connection S^α on the $S^1 \simeq \mathbf{T}_a/\mathbb{Z}_2$ bundle

$$\mathscr{F}^\delta_{\alpha_1,\ldots,\alpha_m,\alpha,\alpha}/\mathbf{T}_{a_1}\times\cdots\times\mathbf{T}_{a_m}\times\mathbf{T}_a \to \mathscr{M}_{\alpha_1,\ldots,\alpha_m,\alpha,\alpha}\,. \tag{4.12}$$

On the other hand, the Atiyah-Singer universal connection on $U^{(m)}$ induces an SO(3)-invariant connection on the bundles $P^{(m)}$ and P defined in Sect. 3. By Proposition 3.20 it follows as in [DW3, Sect. 2.3] that the above connection coincides with S^α. In particular, the connection S^α on (4.12) is SO(3)-invariant.

Let $\imath_{m+1}$, $\hat{\imath}_{m+1}$ denote the inclusions of the generic fiber of π_{m+1}, $\hat{\pi}_{m+1}$ in the generalized Hecke correspondence. Since $\imath_{m+1}$, $\hat{\imath}_{m+1}$ are SO(3)-invariant maps, the connection S^α restricts to an SO(3)-invariant connection on the bundles $\mathscr{O}_{\mathbb{P}^1}(1)$ and $\mathscr{O}_{\mathbb{P}^1}(-1)$, respectively (see also Lemma 3.3 and Proposition 3.13). Therefore, the curvature $\Omega(S^\alpha)$ must be the $\pm$ curvature of the homogeneous connection on $\mathbb{P}^1$, and hence the eigenvalues of the Levi form of $\mathscr{M}_\alpha(0)$ are *uniformly* bounded away from zero. Since the CR-structure is defined everywhere on $\mathscr{M}_\alpha(0)$, we obtain everywhere one positive and one negative eigenvalue. By repeating the same argument for $\imath_{m+2}$, $\hat{\imath}_{m+2}$, the proposition follows. □

Combining Propositions 4.10 and 4.11, we also obtain

Corollary 4.13. *Given* α *a regular value of* r, *there is an* ε_0 *such that for all* t *satisfying* $0 < |t| < \varepsilon_0$ *the Levi form of* $\mathscr{M}_\alpha(t)$ *has everywhere at least two positive and two negative eigenvalues.*

Let us further examine the space $\mathscr{M}_\alpha(0)$ and the map

$$p : \mathscr{M}_\alpha(0) \to \mathscr{M}_{\alpha_1,\ldots,\alpha_m,\alpha,\alpha}\,. \tag{4.14}$$

As in [DW3, Lemma 5.9], p is a principal S^1 CR-bundle, i.e. both the circle action and the map p preserve the CR-structure. Let

$$p : D(\mathscr{M}_\alpha(0)) \to \mathscr{M}_{\alpha_1,\ldots,\alpha_m,\alpha,\alpha}$$

denote the holomorphic disk bundle associate to (4.14) in such a way as to make $\partial D(\mathcal{M}_\alpha(0))$ CR-isomorphic to $\mathcal{M}_\alpha(0)$. We also let $\varepsilon \to \mathcal{M}_{\alpha_1,\dots,\alpha_m,\alpha,\alpha}$ denote the associated line bundle to (4.14). Then it follows from Proposition 3.20 that $\varepsilon \simeq L(0,\dots,0,1,1)$. Recall that $L_\alpha(t)$ denotes the restriction of $\Delta^k(n_1,\dots,n_m)(t)$ to $\mathcal{M}_\alpha(t)$. Set

$$L_\alpha(0) = p^*\Delta^k(n_1,\dots,n_m,0,0)\,,$$

where p is the map (4.14). We have

Proposition 4.15. *Let*

$$\mathfrak{L}_\alpha = \bigcup_{t\in D} L_\alpha(t)\,.$$

Then $\mathfrak{L}_\alpha$ is a continuous family of CR-line bundles parametrized by the disk.

Proof. The argument proceeds exactly as in [DW3, Sect. 6.1].

Proof of Theorem 4.1. By semicontinuity of Kohn-Rossi cohomology (see Corollary 4.13 and [DW3, Theorem 5.8]) we have

$$\dim H^0_{\mathrm{CR}}(\mathcal{M}_\alpha(t), L_\alpha(t)) \leqq \dim H^0_{\mathrm{CR}}(\mathcal{M}_\alpha(0), L_\alpha(0))\,. \tag{4.16}$$

On the other hand, since both the S^1-action and the map p are CR, we have

$$H^0_{\mathrm{CR}}(\mathcal{M}_\alpha(0), L_\alpha(0)) = \bigoplus_{l\in\mathbb{Z}} H^0\left(\mathcal{M}_{\alpha_1,\dots,\alpha_m,\alpha,\alpha}, \Delta^k(n_1,\dots,n_m,0,0)\otimes\varepsilon^l\right)\,. \tag{4.17}$$

Since by Proposition 3.20, $\varepsilon \simeq L(0,\dots,0,1,1)$ we obtain from the vanishing theorem (Theorem 3.8) that

$$\dim H^0_{\mathrm{CR}}(\mathcal{M}_\alpha(0), L_\alpha(0)) = \sum_{l=0}^{k} D^{g,k}_{n_1,\dots,n_m,l,l}\,. \tag{4.18}$$

Combining (4.9), (4.16), and (4.18) proves the theorem. □

4.2 Proof of Theorem 4.2

Now we turn to the proof of Theorem 4.2. Let $\Sigma^+_{p_1,\dots,p_{m^+}}$ and $\Sigma^-_{p_{m^+1},\dots,p_m}$ be compact Riemann surfaces of genus g_+ and g_-, punctured at the points $p_1,\dots,p_{m^+}$ and $p_{m^+1},\dots,p_m$, respectively. We assume $g_+ \geqq 2$ and $g_- \geqq 1$, and let $p \in \Sigma^+_{p_1,\dots,p_{m^+}}$, $p' \in \Sigma^-_{p_{m^+1},\dots,p_m}$ be points with coordinate systems z and z' (see (4.5)). Given a complex number $t \in D^*$, let $\Sigma_{p_1,\dots,p_m}$ denote the Riemann surface of genus $g = g_+ + g_-$ obtained by glueing $\Sigma^+_{p_1,\dots,p_{m^+}}\setminus\{|z| \leqq |t|\}$ and $\Sigma^-_{p_{m^+1},\dots,p_m}\setminus\{|z'| \leqq |t|\}$ via the equation $zz' = t$. Let $\varepsilon = |t|^{1/2}$ and C_ε be the circle in $\Sigma_{p_1,\dots,p_m}(t)$ defined by the equation $|z| = |z'| = \varepsilon$. Let $C^\pm_\varepsilon$ be the circles in $\Sigma^+_{p_1,\dots,p_{m^+}}$ and $\Sigma^-_{p_{m^+1},\dots,p_m}$ defined by $|z| = \varepsilon$, $|z'| = \varepsilon$. We give C_ε the boundary orientation from $\Sigma^+_{p_1,\dots,p_{m^+}}\setminus\{|z| < \varepsilon\}$ which is opposite the one

coming from $\Sigma^-_{p_{m^+1},\dots,p_m}\backslash\{|z'|<\varepsilon\}$. Set

$$\Sigma^+_{p_1,\dots,p_{m^+},p} = \Sigma^+_{p_1,\dots,p_{m^+}}\backslash\{p\}$$

$$\Sigma^-_{p_{m^+1},\dots,p_m,p'} = \Sigma^-_{p_{m^+1},\dots,p_m}\backslash\{p'\}$$

$$\Sigma_{p_1,\dots,p_{m+2}} = \Sigma^+_{p_1,\dots,p_{m^+},p} \cup \Sigma^-_{p_{m^+1},\dots,p_m,p'} ,$$

where we have let $p_{m+1} = p$, $p_{m+2} = p'$. We also fix a family of diffeomorphisms

$$\chi_t : \Sigma_{p_1,\dots,p_{m+2}} \to \Sigma_{p_1,\dots,p_m}(t)\backslash C_\varepsilon ,$$

such that $\chi_0 = id$ and $\chi_t = id$ outside the pinching region $\{|t| < |z|, |z'| < 1\}$.

Notation 4.19. For simplicity we shall use the following notation *for this subsection only*:

$$\Sigma_t = \Sigma_{p_1,\dots,p_m}(t) , \quad \Sigma_0 = \Sigma_{p_1,\dots,p_{m+2}} .$$

Also, given rational weights $\alpha_1,\dots,\alpha_m$ as in (2.1) we set

$$\mathscr{M}(t) = \mathscr{M}_{\alpha_1,\dots,\alpha_m}(t)$$

and

$$r : \mathscr{M}(t) \to [0,1/2] \simeq \mathrm{SU}(2)/\mathrm{SU}(2)_{adj.}$$

for the holonomy map around C_ε. As for the non-separating cycles, Lemma 4.7 holds also in this case.

Given α a regular value of r, which we may assume is also a rational number, set $\mathscr{M}_\alpha(t) = r^{-1}(\alpha)$ with the induced CR-structure S^α defined in (4.8). Following the notation of Sect. 4.1 we let

$$L_\alpha(t) = \Delta^k(n_1,\dots,n_m)(t)|_{\mathscr{M}_\alpha(t)} .$$

Then as in (4.9) we obtain

$$D^{g,k}_{n_1,\dots,n_m} \leqq \dim H^0_{\mathrm{CR}}(\mathscr{M}_\alpha(t), L_\alpha(t)) . \tag{4.20}$$

As in Sect. 4.1, for $[\nabla_t] \in \mathscr{M}(t)$, $T_{[\nabla_t]}\mathscr{M}(t)$ can be identified with the subspace $K_{\nabla_t}(t)$ of $L^2_{-\delta}(T^*\Sigma_t^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}})$, and

$$K_{\nabla_t}(t) \to K_{\nabla_0}(0)$$

$$S^\alpha_{\nabla_t}(t) \to S^\alpha_{\nabla_0}(0)$$

are continuous families of subspaces of $L^2_{-\delta}(T^*\Sigma_t^{0,1} \otimes \mathfrak{g}_E^{\mathbb{C}})$, where $K_{\nabla_0}(0)$ and $S^\alpha_{\nabla_0}(0)$ are defined as in Sect. 4.1. Moreover,

$$S^\alpha_{\nabla_0}(0) \simeq H^1_{\delta^+,\nabla_0^+} \oplus H^1_{\delta^+,\nabla_0^-}$$

can be viewed as a subspace of

$$T_{\nabla_0}(\{\mathscr{F}^{+,\delta}_{\alpha_1,\dots,\alpha_{m^+},\alpha}/\mathbf{T}_{a_1} \times \cdots \times \mathbf{T}_{a_{m^+}}\} \times_{\mathbf{T}_a} \{\mathscr{F}^{-,\delta}_{\alpha_{m^+1},\dots,\alpha_m,\alpha}/\mathbf{T}_{a_{m^+1}} \times \cdots \times \mathbf{T}_{a_m}\}) ,$$

where $\mathbf{T}_a$ acts diagonally. If we define the limiting CR-structure on

$$\mathscr{M}_\alpha(0) = \{\mathscr{F}^{+,\delta}_{\alpha_1,\dots,\alpha_{m^+},\alpha}/\mathbf{T}_{a_1} \times \cdots \times \mathbf{T}_{a_{m^+}}\} \times_{\mathbf{T}_a} \{\mathscr{F}^{-,\delta}_{\alpha_{m^+1},\dots,\alpha_m,\alpha}/\mathbf{T}_{a_{m^+1}} \times \cdots \times \mathbf{T}_{a_m}\}$$

by $S^\alpha(0)$, then we obtain the following analogue of Proposition 4.10.

Proposition 4.21. *Let $\mathfrak{M}_\alpha = \cup_{t\in D}\mathscr{M}_\alpha(t)$. Then $\mathfrak{M}_\alpha$ is a differentiable family of CR-manifolds parametrized by the disk.*

Proposition 4.22. *The Levi form of $\mathscr{M}_\alpha(0)$ has everywhere at least one positive and one negative eigenvalue. If we also assume $g_- \geqq 2$, then there are at least two eigenvalues of each sign.*

Proof. As in [DW3, Theorem 4.16], the Levi form of $\mathscr{M}_\alpha(0)$ equals the direct sum of the curvatures $\Omega(S^\alpha_+) \oplus \Omega(S^\alpha_-)$, where $S^\alpha_\pm$ are the connections induced on the fibrations

$$\mathscr{F}^{+,\delta}_{\alpha_1,\dots,\alpha_{m^+},\alpha}/\mathbf{T}_{a_1} \times \cdots \times \mathbf{T}_{a_{m^+}} \to \mathscr{M}^+_{\alpha_1,\dots,\alpha_{m^+},\alpha}$$

$$\mathscr{F}^{-,\delta}_{\alpha_{m^+1},\dots,\alpha_m,\alpha}/\mathbf{T}_{a_{m^+1}} \times \cdots \times \mathbf{T}_{a_m} \to \mathscr{M}^-_{\alpha_{m^+1},\dots,\alpha_m,\alpha}\,.$$

Since $g_+ \geqq 2$ we can restrict $\Omega(S^\alpha_+)$ to the fibers of the Hecke correspondence associated to the Riemann surface $\Sigma^+_{p_1,\dots,p_{m^+},p}$ and the puncture p. Thus we obtain as in Proposition 4.11 one positive and one negative eigenvalue. The second statement is clear, since then we may do the same with S^α_-. □

Combining Proposition 4.21 and 4.22 we obtain the following analogue of Corollary 4.13.

Corollary 4.23. *Given α a regular value of r, there is an ε_0 such that for all t satisfying $0 < |t| < \varepsilon_0$ the Levi form of $\mathscr{M}_\alpha(t)$ has everywhere at least one positive and one negative eigenvalue.*

Next we examine $\mathscr{M}_\alpha(0)$ and the map

$$p : \mathscr{M}_\alpha(0) \to \mathscr{M}^+_{\alpha_1,\dots,\alpha_{m^+},\alpha} \times \mathscr{M}^-_{\alpha_{m^+1},\dots,\alpha_m,\alpha}\,.$$

As in [DW3, Lemma 5.9] and Sect. 4.1, p is a principal S^1 CR-bundle. Let $D(\mathscr{M}_\alpha(0))$ and ε denote the associated complex disk bundle and line bundle, respectively. It follows from Proposition 3.15 that ε is isomorphic to $L^+(0,\dots,0,2) \otimes L^-(0,\dots,0,2)$. Recall that $L_\alpha(t)$ is the restriction of $\Delta^k(n_1,\dots,n_m)$ to $\mathscr{M}_\alpha(t)$. We now make the assumption that the integers are chosen such that $\sum_{i=1}^{m^+} n_i \in 2\mathbb{Z}$. The case where this sum is odd can also be treated, and we shall state our results on the dimensions (e.g. (4.26)) without such an assumption. Set

$$L_\alpha(0) = \pi^*(\Delta^k_+(n_1,\dots,n_{m^+},0) \otimes \Delta^k_-(n_{m^+1},\dots,n_m,0))\,.$$

Then we have as in Proposition 4.15:

Proposition 4.24. *Let*

$$\mathfrak{L}_\alpha = \bigcup_{t \in D} L_\alpha(t) .$$

Then $\mathfrak{L}_\alpha$ is a continuous family of CR-line bundles parametrized by the disk.

Proof of Theorem 4.2. By semicontinuity of Kohn-Rossi cohomology, inequality (4.16) still holds. The isomorphism (4.17) becomes in this case

$$(4.25) \qquad H^0_{\mathrm{CR}}(\mathscr{M}_\alpha(0), L_\alpha(0)) = \bigoplus_{l \in 2\mathbb{Z}} H^0(\mathscr{M}^+_{\alpha_1,\ldots,\alpha_{m^+},\alpha}, \Delta^k_+(n_1,\ldots,n_{m^+},l)) \otimes H^0(\mathscr{M}^-_{\alpha_{m^+1},\ldots,\alpha_m,\alpha}, \Delta^k_-(n_{m^+1},\ldots,n_m,l))$$

By the vanishing theorem for the first factor in the tensor product, (4.25) becomes

$$(4.26) \qquad \dim H^0_{\mathrm{CR}}(\mathscr{M}_\alpha(0), L_\alpha(0)) = \sum_{l=0}^{k} D^{g_+,k}_{n_1,\ldots,n_{m^+},l} D^{g_-,k}_{n_{m^+1},\ldots,n_m,l} .$$

Combining (4.16), (4.20), and (4.26) proves Theorem 4.2. □

Finally, we would like to point out that in the case of no punctures ($m = 0$), we have previously shown [DW3, Main Theorem]

Theorem 4.27. *Assume $g_\pm \geqq 2$. Then*

$$D^{g,k} = \sum_{l=0}^{k} D^{g_+,k}_l D^{g_-,k}_l .$$

Proof. In the proof given in [DW3], it was assumed that $g_\pm \geqq 4$. This was because the vanishing theorem (Theorem 3.8) had only been proven by using topological considerations which required high codimension (see [DW2]). However, we have shown in Theorem 3.8 that the vanishing holds under the weaker assumption $g_\pm \geqq 2$. All the arguments then go through to prove the result Theorem 4.27. □

5 Proof of the main Theorem

To complete the proof of the Verlinde formula Theorem 1.4, we must first show that the inequalities in Theorems 4.1 and 4.2 are in fact equalities. Then we can apply the known factorization formulas for $N^{g,k}_{n_1,\ldots,n_m}$ to show inductively that if $D^{g,k}_{n_1,\ldots,n_m} = N^{g,k}_{n_1,\ldots,n_m}$ for genus 2 with any choice of $n_1,\ldots,n_m$, then equality holds for all $g \geqq 2$. We shall do this in steps. First, in Sect. 5.1 we shall show that it suffices to consider just one puncture (see Proposition 5.9). Then in Sect. 5.2 shall show how the main theorem of [DW3] (Theorem 4.27 above) and the inequalities of Theorems 4.1 and 4.2 actually prove equalities.

5.1 Induction on the number of punctures

Recall that for small α,

$$\mathcal{M}_{\alpha_1,\dots,\alpha_m,\alpha} \simeq \mathbb{P}(U_{p_{m+1}}^{(m)*}), \tag{5.1}$$

where $U_{p_{m+1}}^{(m)*}$ is the dual universal bundle on $\Sigma \times \mathcal{M}_{\alpha_1,\dots,\alpha_m}$ restricted to the point $\{p_{m+1}\}$ (see Sect. 2.4 and the discussion preceding Lemma 3.10). Let $\pi = \pi_{m+1}$ be the "puncture forgetting" map (2.14), which in this case is simply the bundle projection in (5.1). Let $\mathcal{O}(-1)$ denote the tautological line bundle for the projectivization in (5.1), and as usual let $\mathcal{O}(1)$ denote its dual and $\mathcal{O}(l) = \mathcal{O}(1)^{\otimes l}$. Then we have

Lemma 5.2. *For any choice of integers $n_1,\dots,n_m,n_{m+1}$ with even sum,*

$$L(n_1,\dots,n_m,n_{m+1}) \simeq \pi^* L(n_1 - n_{m+1}, n_2,\dots,n_m) \otimes \mathcal{O}(n_{m+1}).$$

Proof. Recall from Sect. 2.4 that the universal bundle $U^{(m)}$ is obtained by the quotient of $\tilde{U} \otimes \tilde{L}_1$ on $\Sigma \times \mathcal{A}_{p.s.}$. Set $p = p_{m+1}$. If we restrict to $\{p\} \times \mathcal{A}_{p.s.}$ and check the action of $\mathcal{G}_{par.}^{\mathbb{C}}$ on the tautological line in $\tilde{U}|_{\{p\}} \otimes \tilde{L}_1$, we see that $\mathcal{O}(1)$ is holomorphically isomorphic to $L(1,0,\dots,0,1)$. The result now follows by noting that $L(n_1,\dots,n_m,0)$ on $\mathcal{M}_{\alpha_1,\dots,\alpha_m,\alpha}$ is isomorphic to $\pi^* L(n_1,\dots,n_m)$ on $\mathcal{M}_{\alpha_1,\dots,\alpha_m}$. □

The main consequence of this is the following

Proposition 5.3. *Suppose $1 \leqq n_j \leqq k$, $j = 2,\dots,m$ and $1 \leqq n_1 < k$. Suppose also that the sum of $n_1,\dots,n_m$ is odd. Then*

$$H^0(\mathcal{M}_{\alpha_1,\dots,\alpha_m,\alpha}, \Delta^k(n_1,\dots,n_m,1)) \simeq H^0(\mathcal{M}_{\alpha_1,\dots,\alpha_m}, \Delta^k(n_1 - 1, n_2,\dots,n_m) \oplus H^0(\mathcal{M}_{\alpha_1,\dots,\alpha_m}, \Delta^k(n_1 + 1, n_2,\dots,n_m)).$$

Proof. By Lemma 5.2 we have

$$H^0(\mathcal{M}_{\alpha_1,\dots,\alpha_m,\alpha}, \Delta^k(n_1,\dots,n_m,1)) \simeq H^0(\mathcal{M}_{\alpha_1,\dots,\alpha_m}, \Delta^k(n_1 - 1, n_2,\dots,n_m) \otimes U_p^{(m)}). \tag{5.4}$$

We shall henceforth drop the $\mathcal{M}_{\alpha_1,\dots,\alpha_m}$ argument when it is understood. Let $L(2) = L(2,0,\dots,0)$ denote the quotient of $\tilde{L}_1^{\otimes 2}$ on $\mathcal{M}_{\alpha_1,\dots,\alpha_m}$. Note that $L(2)$ is *not* the square of a line bundle. Then we have the following

Lemma 5.5. *On $\mathcal{M}_{\alpha_1,\dots,\alpha_m}$ we have the exact sequence*

$$0 \to \mathcal{O} \to U_{p_1}^{(m)} \to L(2) \to 0.$$

Lemma 5.6. *There exists a choice of generic weights $\alpha_1,\dots,\alpha_m$ such that both*

$$H^1(\mathcal{M}_{\alpha_1,\dots,\alpha_m}, \Delta^k(n_1 - 1, n_2,\dots,n_m)) = 0,$$

$$H^1(\mathcal{M}_{\alpha_1,\dots,\alpha_m}, \Delta^k(n_1 + 1, n_2,\dots,n_m)) = 0.$$

Assuming these two lemmas, the proposition follows immediately by tensoring the exact sequence in Lemma 5.5 with $\Delta^k(n_1-1,n_2,\ldots,n_m)$ and letting $p \to p_1$ in (5.4). The vanishing from Lemma 5.6 guarantees that the dimension of H^0 does not jump. □

Proof of Lemma 5.5. Consider the universal bundle $\tilde{U} \to \Sigma \times \mathscr{A}_{p.s.}$ and its restriction $\tilde{U}_{p_1}$ to $\{p_1\} \times \mathscr{A}_{p.s.}$. Then the flag at p_1 induces a natural map $\tilde{U}_{p_1} \to \tilde{L}_1 \to 0$. Indeed, given $\bar{\partial}_E \in \mathscr{A}_{p.s.}$ a flag ℓ at p_1 defines a surjection $E_{p_1}^{\bar{\partial}_E} \to \mathbb{C} \to 0$, which extends to a map

$$\tilde{U}_{p_1} \xrightarrow{\ell} \mathscr{A}_{p.s.} \times \mathbb{C} \to 0\,. \tag{5.5}$$

If φ is a local equivariant section of $\tilde{U}_{p_1}$ and $g \in \mathscr{G}_{par.}^{\mathbb{C}}$, then

$$\ell \circ \varphi(g \cdot \bar{\partial}_E) = \ell \circ (g \cdot \varphi)(\bar{\partial}_E) = g(p_1) \cdot \ell \circ \varphi(\bar{\partial}_E) = \chi_1(g)\ell \circ \varphi(\bar{\partial}_E)\,,$$

where χ_1 is the character defined by evaluation at p_1 and the fundamental representation. Hence, $\ell \circ \varphi$ is a section of $\tilde{L}_1$. The kernel of (5.5) is the line bundle on $\mathscr{A}_{p.s.}$ defined by the kernel of ℓ in $\tilde{U}_{p_1}$. Clearly, this can be identified with $\tilde{L}_1^*$. Thus on $\mathscr{A}_{p.s.}$ we have

$$0 \to \tilde{L}_1^* \to \tilde{U}_{p_1} \to \tilde{L}_1 \to 0\,.$$

Tensoring by $\tilde{L}_1$ and noting that the resulting sequence is $\mathscr{G}_{par.}^{\mathbb{C}}$ equivariant proves the result. □

Proof of Lemma 5.6. By [B1, Corollary 3.5] there exist weights $\alpha_1,\ldots,\alpha_m$ and $\alpha_1',\ldots,\alpha_m$ such that both

$$\Delta^k(n_1-1,n_2,\ldots,n_m)^* \otimes K \to \mathscr{M}_{\alpha_1,\ldots,\alpha_m}\,,$$

$$\Delta^k(n_1+1,n_2,\ldots,n_m)^* \otimes K' \to \mathscr{M}_{\alpha_1',\ldots,\alpha_m}\,,$$

are negative, where K and K' denote the canonical bundles on $\mathscr{M}_{\alpha_1,\ldots,\alpha_m}$ and $\mathscr{M}_{\alpha_1',\ldots,\alpha_m}$, respectively. The case where $m=1$ was proven in [DW3, Theorem 6.9]. For the case $m=2$, $\Delta^k(n_1,n_2)^* \otimes K_{\mathscr{M}_{\alpha_1,\alpha_2}}$ is always semi-negative and is negative on a dense, open set, regardless of the weights. Hence, we may assume $m \geqq 3$ (cf. [S]). Strictly speaking, the particular choices $\alpha_1,\ldots,\alpha_m$ and $\alpha_1',\ldots,\alpha_m$ given in [B1] may be non-generic. Making α_1 a bit larger and α_1' a bit smaller guarantees that the weights are generic, but now the line bundles above can only be assumed to be semi-negative and negative on a dense open set. In any event, again referring to [B1] one can see that there is at most one non-generic value of $\alpha, \alpha_2,\ldots,\alpha_m$ for $\alpha_1 < \alpha < \alpha_1'$. Consequently, it follows from standard techniques (cf. [BH, Sect. 5.1]) that $\mathscr{M}_{\alpha_1,\ldots,\alpha_m}$ and $\mathscr{M}_{\alpha_1',\ldots,\alpha_m}$ are related by a simple modification Z:

$$\begin{array}{ccc} & Z & \\ {\scriptstyle\pi}\swarrow & & \searrow{\scriptstyle\pi'} \\ \mathscr{M}_{\alpha_1,\ldots,\alpha_m} & & \mathscr{M}_{\alpha_1',\ldots,\alpha_m} \end{array} \tag{5.7}$$

where Z is the mutual blow-up of $\mathcal{M}_{\alpha_1,\dots,\alpha_m}$ and $\mathcal{M}_{\alpha'_1,\dots,\alpha_m}$ along smooth centers X and X' which are projective bundles over the Jacobian variety of Σ. Then a calculation as in [BH] shows:

$$\operatorname{codim}(X) + \operatorname{codim}(X') = 2g - 2 + m\,.$$

Hence, we may assume without loss of generality that $\operatorname{codim}(X') \geqq 3$. Let

$$\mathcal{L} = \Delta^k(n_1+1, n_2, \dots, n_m)^* \otimes K \to \mathcal{M}_{\alpha_1,\dots,\alpha_m}\,,$$

$$\mathcal{L}' = \Delta^k(n_1+1, n_2, \dots, n_m)^* \otimes K' \to \mathcal{M}_{\alpha'_1,\dots,\alpha_m}\,.$$

By the Nakano vanishing theorem for the semi-negative case [S], $H^{n-1}(\mathcal{L}') = 0$, where $n = \dim \mathcal{M}_{\alpha_1,\dots,\alpha_m}$. The lemma is proven if we can show also $H^{n-1}(\mathcal{L}) = 0$. Let A denote the exceptional divisor in (5.7), and set $\tilde{\mathcal{L}} = \pi^*\mathcal{L}$, $\tilde{\mathcal{L}}' = (\pi')^*\mathcal{L}'$. Then $\tilde{\mathcal{L}}' = \tilde{\mathcal{L}} \otimes A^\mu$ for some $\mu \geqq 0$. We claim that also $\mu \leqq 1$. For let $L(2)$ be as above. Then the assumption on the weights implies that $\mathcal{L} \otimes L(2)^{-1}$ is semi-negative, and so $\tilde{\mathcal{L}} \otimes \pi^* L(2)^{-1}$ is semi-negative. Restricting to the fiber of π' implies that $\mu - c \leqq 0$, where c is the Chern class of $L(2)|_X$. But this is easily seen to be 1. If $\mu = 0$, then we are finished. Otherwise, suppose $\mu = 1$. Then we have an exact sequence

$$0 \to \tilde{\mathcal{L}} \to \tilde{\mathcal{L}}' \to \tilde{\mathcal{L}}'|_A \to 0\,,$$

and a corresponding long exact sequence

$$\cdots \to H^{n-2}(\tilde{\mathcal{L}}'|_A) \to H^{n-1}(\tilde{\mathcal{L}}) \to H^{n-1}(\tilde{\mathcal{L}}') \to \cdots$$

But since $H^{n-1}(\tilde{\mathcal{L}}') = H^{n-1}(\mathcal{L}') = 0$, and $\tilde{\mathcal{L}}'|_A$ is pulled back from X' which has dimension $\leqq n-3$, we must also have $H^{n-2}\left(\tilde{\mathcal{L}}'|_A\right) = 0$. This completes the proof of Lemma 5.6. □

Recall from Theorem 3.8 that $D^{g,k}_{n_1,\dots,n_m} = \dim \mathcal{V}^{g,k}_{n_1,\dots,n_m}$ is independent of the choice of $\alpha_1,\dots,\alpha_m$. Note that Proposition 5.3 implies

$$D^{g,k}_{n_1,\dots,n_m,1} = D^{g,k}_{n_1+1,n_2,\dots,n_m} + D^{g,k}_{n_1-1,n_2,\dots,n_m}\,. \tag{5.8}$$

This gives the main result of this subsection:

Proposition 5.9. *Fix $g \geqq 2$. Assume that Theorem 1.4 is true for genus g with one puncture ($m = 1$) and any choice of integer n_1. Then it is true for genus g with an arbitrary number of punctures and arbitrary weights $n_1,\dots,n_m$.*

Proof. The following argument is very similar to that of Gawędzki and Kupiainin for the case of genus zero [GK]. The numbers $N^{g,k}_{n_1,\dots,n_m}$ defined in (1.2) satisfy

$$N^{g,k}_{n_1,\dots,n_m,a,b} = \sum_{l=0}^{k} N^{g,k}_{n_1,\dots,n_m,l} N_{l,a,b}\,,$$

where the $N_{l,a,b}$ (the "fusion rules") are the truncated Clebsch-Gordon coefficients (see [V], [MoSb])

$$N_{l,a,b} = \begin{cases} 1 & \text{if } l = |a-b|, |a-b|+2, \ldots, \min\{a+b, 2k-a-b\}, \\ 0 & \text{otherwise.} \end{cases}$$

Then (5.8) holds with D's replaced by N's, since the $N_{l,a,b}$ satisfy

$$N_{l,1,b} = \begin{cases} 1 & \text{if } l = b-1 \geqq 0 \text{ or } l = b+1 \leqq k, \\ 0 & \text{otherwise.} \end{cases} \tag{5.10}$$

Hence, given (5.8) and assuming that for all n_1, $D^{g,k}_{n_1} = N^{g,k}_{n_1}$, then by induction on the number of points, $D^{g,k}_{n_1,1,\ldots,1} = N^{g,k}_{n_1,1,\ldots,1}$ for any n_1 and any number of 1's. Now suppose we have proven that $D^{g,k}_{n_1,\ldots,n_m} = N^{g,k}_{n_1,\ldots,n_m}$ for any m and integers $0 \leqq n_1, \ldots, n_m \leqq l < k$. Consider $D^{g,k}_{n_1,\ldots,n_m}$ where some n_j, say $n_1 = l+1 \geqq 1$. Then we have

$$\begin{aligned} D^{g,k}_{n_1,\ldots,n_m} &= D^{g,k}_{n_1-1,n_2,\ldots,n_m,1} - D^{g,k}_{n_1-2,n_2,\ldots,n_m} && \text{by (5.8)} \\ &= N^{g,k}_{n_1-1,n_2,\ldots,n_m,1} - N^{g,k}_{n_1-2,n_2,\ldots,n_m} && \text{by assumption} \\ &= N^{g,k}_{n_1,\ldots,n_m} && \text{by (5.10).} \end{aligned}$$

By induction, this completes the proof. □

5.2 Induction on the genus

The goal of this subsection is to prove the following

Proposition 5.11. *The Verlinde formula is true for genus $g \geqq 2$ with one puncture and any integer n_1, i.e. $D^{g,k}_{n_1} = N^{g,k}_{n_1}$ for all n_1.*

As a consequence of Propositions 5.9 and 5.11, we have a proof of the general Verlinde formula for SU(2), Theorem 1.4.

We prove Proposition 5.11 by a simple induction on the genus. The first step is genus 2.

Lemma 5.12. *For any choice of integers $n_1, \ldots, n_m$, $D^{2,k}_{n_1,\ldots,n_m} = N^{2,k}_{n_1,\ldots,n_m}$.*

Proof. By Proposition 5.9 it suffices to prove the result for one puncture. This may be done by using the explicit description of the moduli space due to Newstead. We defer the details to the Appendix (Proposition A.1). □

Lemma 5.13. *Suppose that for fixed $g \geqq 2$, $D^{g,k}_{n_1} = N^{g,k}_{n_1}$ for all n_1. Then $D^{g+2,k} = N^{g+2,k}$.*

Proof. From the main theorem of [DW3] (Theorem 4.27 above),

$$D^{g+2,k} = \sum_{l=0}^{k} D^{g,k}_l D^{2,k}_l = \sum_{l=0}^{k} N^{g,k}_l N^{2,k}_l = N^{g+2,k}$$

where the second equality holds by assumption and Lemma 5.12, and the third equality holds since the N's satisfy the factorization rules (4.4). □

Lemma 5.14. *Under the same assumptions as in Lemma* 5.13, *we have* $D_{n_1}^{g+1,k} = N_{n_1}^{g+1,k}$ *for all* n_1.

Proof. By Theorem 4.1 we have

$$D_{n_1}^{g+1,k} \leqq \sum_{l=0}^{k} D_{n_1,l,l}^{g,k} .$$

Now by the assumption and Proposition 5.9, we have $D_{n_1,l,l}^{g,k} = N_{n_1,l,l}^{g,k}$ for all n_1 and l. Hence,

$$D_{n_1}^{g+1,k} \leqq \sum_{l=0}^{k} N_{n_1,l,l}^{g,k} = N_{n_1}^{g+1,k} , \tag{5.15}$$

since the N's satisfy the factorization rules (4.3). It is not hard to see that the moduli of parabolic bundles on the torus with one puncture is isomorphic to $\mathbb{P}^1$. It then follows from direct computation (cf. proof of Theorem 3.8) that $D_l^{1,k} = N_l^{1,k} = k - l + 1$ for all k and even l satisfying $0 \leqq l \leqq k$. Using this fact, Theorem 4.2, (5.15), and (4.3), we obtain

$$D^{g+2,k} \leqq \sum_{l=0}^{k} D_l^{g+1,k} D_l^{1,k} \leqq \sum_{l=0}^{k} N_l^{g+1,k} N_l^{1,k} = N^{g+2,k} .$$

But now Lemma 5.13 implies all the inequalities must in fact be equalities. □

By induction, Lemmas 5.12 and 5.14 imply Proposition 5.11, and this completes the proof of Theorem 1.4. □

Appendix – Genus 2 with one puncture

In this appendix, we prove the Verlinde formula for genus 2 and one puncture. This was the inductive step necessary in Sect. 5.2 (see Lemma 5.12). Fortunately, Newstead has given an explicit description of the moduli space in question [N1] – it is a projective bundle over the intersection of two quadrics in $\mathbb{P}^5$. In particular, the cohomology ring is completely known, and the Verlinde formula is a consequence of Riemann-Roch.

Proposition A.1. *For all integers* n, $D_n^{2,k} = N_n^{2,k}$.

Proof. The proof consists of writing out both sides and verifying that they are equal. We begin by computing the dimension of the space of sections according to the Riemann-Roch formula. Let $U_p \to \mathcal{M}(2,-1)$ be the universal bundle restricted to a point p, normalized such that $c_1(U_p) = \alpha$ is the positive generator of $H^2(\mathcal{M}(2,-1),\mathbb{Z})$. For a given even integer n, the line bundle we are interested in is $L_n = (\pi^*\alpha)^{n/2} \otimes \xi^{k-n}$, where ξ is the $\mathcal{O}(1)$ on $\pi : \mathbb{P}(U_p^*) \to \mathcal{M}(2,-1)$. Since for $0 \leqq n \leqq k$ the higher cohomology of L_n vanishes (see [DW3, Theorem 6.9]) as well as its higher direct images under π, it suffices to compute the Euler characteristic of $\pi_* L_n = \alpha^{n/2} \otimes S^{k-n} U_p$. We need the following

Lemma A.2. *Let U be a rank 2 bundle, $\alpha = c_1(U)$, $\beta = -c_2(\mathrm{End}\, U)$. Then we have for all $l \in \mathbf{Z}^+$:*

$$\mathrm{ch}(S^l U) = \exp(l\alpha/2)\frac{\sinh\left(\frac{l+1}{2}\sqrt{\beta}\right)}{\sinh(\sqrt{\beta}/2)}.$$

Proof. By the splitting principle it suffices to check this when $U = A \oplus B$, where A and B are line bundles. Let $a = c_1(A)$, $b = c_1(B)$. Then

$$\mathrm{ch}(S^l U) = \sum_{j=0}^{l} \mathrm{ch}(A^j \otimes B^{(l-j)}) = \sum_{j=0}^{l} e^{ja} \cdot e^{(l-j)b} = e^{lb} \sum_{j=0}^{l} e^{j(a-b)}.$$

Let $x = (a-b)/2 = \sqrt{\beta}/2$. We must show that the right hand side above is equal to

$$e^{l(a+b)/2} \cdot \frac{\sinh(l+1)x}{\sinh x} = e^{lb} \cdot e^{lx} \cdot \frac{\sinh(l+1)x}{\sinh x},$$

or that

$$e^{lx} \cdot \frac{\sinh(l+1)x}{\sinh x} = \sum_{j=0}^{l} e^{2jx}.$$

But the left hand side is

$$e^{lx} \cdot \frac{\sinh(l+1)x}{\sinh x} = e^{lx} \cdot \frac{e^{(l+1)x} - e^{-(l+1)x}}{e^x - e^{-x}} = 1 + e^{2x} + \cdots + e^{2xl}. \qquad \square$$

From the lemma we have

$$\mathrm{ch}(\pi_* L_n) = e^{k\alpha/2} \cdot \frac{\sinh\left(\frac{k-n+1}{2}\sqrt{\beta}\right)}{\sinh\sqrt{\beta}/2}.$$

We also have (cf. [T1])

$$\mathrm{td}(\mathscr{M}(2,-1)) = \exp(\alpha)\left(\frac{\sqrt{\beta}/2}{\sinh\sqrt{\beta}/2}\right)^2,$$

in genus 2. Hence, by Grothendieck-Riemann-Roch,

$$\chi(L_n) = \int_{\mathscr{M}(2,-1)} e^{(k+2)\alpha/2} \cdot \frac{\sinh\left(\frac{k-n+1}{2}\sqrt{\beta}\right)}{\sinh\sqrt{\beta}/2}\left(\frac{\sqrt{\beta}/2}{\sinh\sqrt{\beta}/2}\right)^2.$$

Using the expansion of $\sinh x$ we obtain

$$\chi(L_n) = \frac{(k-n+1)(k+2)}{12}\left\{(k+2)^2\frac{\alpha^3}{4} + ((k-n+1)^2 - 3)\frac{\alpha\beta}{4}\right\}[\mathscr{M}(2,-1)].$$

In genus 2, β is half the first Pontryagin class $\beta = (c_1^2 - 2c_2)/2 = -\alpha^2$ (see [N2], p. 261). Moreover, since $\mathscr{M}(2,-1)$ is the intersection of two quadrics in

$\mathbb{P}^5$, the degree $\alpha^3[\mathscr{M}(2,-1)] = 4$. Hence,

(A.4)

$$D_n^{2,k} = \dim \Gamma(\mathscr{M}_\alpha, L_n) = \frac{(k-n+1)(k+2)}{12}\left\{(k+2)^2 - (k-n+1)^2 + 3\right\}.$$

We now show that (A.4) is in accord with the Verlinde formula. Let S_{nm} be as in (1.1). Note that $S_{(k-n)m} = (-1)^m S_{nm}$. We have (see (1.2))

$$N_n^{2,k} = \sum_{m=0}^{k} \frac{S_{nm}}{(S_{m0})^3} = \sum_{m=0}^{k} (-1)^m \frac{S_{(k-n)m}}{(S_{m0})^3}$$

$$= \frac{k+2}{2} \sum_{m=0}^{k} (-1)^m \frac{\sin\dfrac{(k-n+1)(m+1)\pi}{k+2}}{\left(\sin\dfrac{(m+1)\pi}{k+2}\right)^3}.$$

We assume that n is even, since otherwise $D_n^{2,k} = N_n^{2,k} = 0$. Following [Z], we can therefore rewrite this as

$$N_n^{2,k} = (k+2) \sum_{\substack{\xi^{2k+4}=1 \\ \xi \neq \pm 1}} \xi^{k+2} \cdot \frac{\xi^{k-n+1} - \xi^{-(k-n+1)}}{(\xi - \xi^{-1})^3}$$

$$= (k+2) \sum_{\substack{\xi^{2k+4}=1 \\ \xi \neq \pm 1}} \xi^{k+2}\xi^{-(k-n-2)} \cdot \frac{1 - \xi^{2(k-n+1)}}{(1-\xi^2)^3}$$

$$= 2(k+2) \sum_{\substack{\xi^{k+2}=1 \\ \xi \neq 1}} \xi^{(k+2)/2}\xi^{-(k-n-2)/2} \cdot \frac{1 - \xi^{k-n+1}}{(1-\xi)^3}$$

$$= 2(k+2) \sum_{\substack{\xi^{k+2}=1 \\ \xi \neq 1}} \operatorname*{res}_{z=\xi} \left\{ \frac{z^{(k+2)/2} z^{-(k-n-2)/2}(1 - z^{k-n+1})(k+2)}{(z^{k+2}-1)(1-z)^3} \frac{dz}{z} \right\}.$$

Letting $z = e^{2x}$ and applying the residue theorem, we find

$$N_n^{2,k} = -\frac{k+2}{2} \operatorname*{res}_{x=0} \left\{ \frac{(k+2)\sinh[(k-n+1)x]}{\sinh[(k+2)x](\sinh x)^3} dx \right\}$$

$$= -\frac{k+2}{2} \operatorname*{coef}_{x^3} \left\{ \frac{(k+2)x}{\sinh[(k+2)x]} \left(\frac{x}{\sinh x}\right)^3 \sinh[(k-n+1)x] \right\}.$$

Finally, expanding the sinh's, we obtain

$$N_n^{2,k} = -\frac{k+2}{2} \left\{ \frac{(k-n+1)^3}{6} - \frac{k-n+1}{2} - \frac{(k+2)^2}{6}(k-n+1) \right\}$$

$$= \frac{(k-n+1)(k+2)}{12} \left\{(k+2)^2 - (k-n+1)^2 + 3\right\},$$

which agrees with (A.4). This proves Proposition A.1. □

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