Local a Posteriori Error Estimates and Adaptive Control of Pollution Effects

Xiaohai Liao, Ricardo H. Nochetto

Abstract

Local a posteriori error estimators are derived for linear elliptic problems over general polygonal domains in $2d$. The estimators lead to a sharp upper bound for the energy error in a local region of interest. This upper bound consists of H^1-type local error indicators in a slightly larger subdomain, plus weighted L^2-type local error indicators outside this subdomain which account for the pollution effects. This constitutes the basis of a local adaptive refinement procedure. Numerical experiments show a superior performance than the standard global procedure as well as the generation of locally quasi-optimal meshes.

Key words. Linear elliptic problems, finite elements, local a posteriori error estimators, adaptive mesh refinement, quasi-optimal meshes

1 Introduction

Adaptive finite element methods to solve partial differential equations numerically are widely used in science and engineering. One of the main advantages of adaptivity is that it provides a meaningful approach towards multiscale phenomena. The key ingredient of adaptivity is a posteriori error estimators. They are computable estimates for the error in a suitable norm in terms of the approximate solution and the data of the problem. The typical implemention of a posteriori error estimators consists of splitting them into element indicators, which are merely element contributions, and then employing them to make judicious mesh modifications. The ultimate purpose is to equidistribute the approximation errors on the whole domain, which pressumably leads to quasi-optimal meshes. We refer to [2, 22] for references.

Standard a posteriori error estimators are typically *global*, and are thus effective to compute a globally accurate discrete solution with nearly minimal computational resources. In many practical problems, however, the phenomenon of interest is much smaller than the typical problem size, and hence it is critical to concentrate the computational effort to guarantee *local* accuracy in a preassigned region; see [4, 5, 12] and references cited therein. It is known though that this entails coping with the so-called *pollution effect* from the rest of the domain, namely the lack of accuracy due to insufficient resolution of solution features (singularities, internal or boundary layers, etc) outside the local region. This paper addresses how to account for and quantify the pollution effect in the energy norm via *local a posteriori* error estimators.

There are already several papers and books dealing with this important issue. A short discussion is given in [2] indicating that the sum of the H^1 residual-type indicators over a subdomain does not measure the error over the subdomain. Papers [4, 5, 12] do not provide a fully rigorous approach applicable to elliptic PDEs. The rigorous results of [24] do not contain a *sharp computable* estimator to handle the pollution effect of reentrant corners. On the other hand, the rigorous results of [6, 7], based on duality arguments, apply to *linear functionals* of the discrete solution but not to the energy error. Our approach, instead, is based on the fundamental idea of Nitsche and Schatz [17]: the local error in energy norm can be estimated by the local interpolation error plus a global error in a weaker (perhaps negative) norm. We give an a posteriori version of this a priori result.

To describe our main result, consider the following scalar elliptic equation:

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = f, & \text{in } \Omega, \\ \qquad\qquad\quad u = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded polygonal domain, possibly with reentrant corners, f is a given function in $L^2(\Omega)$, and the coefficient matrix $A(x)$ is symmetric positive definite and smooth. Let $\mathcal{T}_h = \{T\}$ be a conforming shape-regular triangulation of Ω with local mesh size h. Let u_h be the finite element solution to (1.1) in some conforming finite element space $\mathring{S}_h(\Omega)$. Let Ω_0 indicate the subdomain of interest of Ω, and let $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$ and $d = \operatorname{dist}(\partial\Omega_0\backslash\partial\Omega, \partial\Omega_1\backslash\partial\Omega)$. The *local a priori* error estimates of Nitsche and Schatz read as follows [17]:

$$\|u-u_h\|_{1,\Omega_0} \le C_1 \inf_{v\in\mathring{S}_h(\Omega)} \|u-v\|_{1,\Omega_1} + \frac{C_2}{d^{1+j}}\|u-u_h\|_{-j,\Omega_1}, \tag{1.2}$$

with $j \ge 0$; $C_1, C_2 > 0$ are interpolation constants. The pollution effect of diffusion from outside Ω_0 is hidden in the negative norm $\|u-u_h\|_{-j,\Omega}$.

Our main goal in this paper is to derive a computable version of (1.2) and use it to drive an adaptive local refinement procedure. To this end, we let $\eta_1(\Omega_1)$ be the usual H^1 residual-type error indicators in the subdomain Ω_1, and $\eta_{0,-\beta}(\Omega\backslash\Omega_1)$ be a weighted L^2 residual-type error indicator over the complement $\Omega\backslash\Omega_1$ with a weight roughly $r^{-\beta_i}$ near the i-th reentrant corner; see Section 2. In Section 3 we prove the *local a posteriori* error estimates for nonconvex domains:

$$\|u-u_h\|_{1,\Omega_0} \le C_1\eta_1(\Omega_1) + \frac{C_2}{d}\max_{1\le i\le I}|\log h_i|^{1/2}\ \eta_{0,-\beta}(\Omega\backslash\Omega_1); \tag{1.3}$$

$C_1, C_2 > 0$ are again interpolation constants and I is the total number of reentrant corners. Since $\eta_{0,-\beta}(\Omega\backslash\Omega_1)$ is of higher order than $\eta_1(\Omega\backslash\Omega_1)$, especially when the size of Ω_1 is much smaller than that of Ω, we could expect that an adaptive refinement strategy based on (1.3) would be more efficient in achieving a desirable local accuracy over Ω_0 with nearly minimal total number of degrees of freedom. This is illustrated experimentally in Section 4 via several examples. The estimate (1.3) does not provide global error control though. For a smooth domain and higher order elements, the second term in the right hand side of (1.3) can be replaced by a higher order estimator (see (3.29), (3.32)). This result conforms with the well known fact that higher order finite elements behave more locally. This is further confirmed experimentally in Section 4.

An outline of this paper is as follows. In Section 2, we collect some preliminary results, notations, and basic assumptions. In Section 3, we derive the local a posteriori error estimates (1.3) along with some variants for conforming finite elements. In Section 4, we propose an adaptive refinement strategy based on (1.3) and report some numerical experiments computed with the finite element toolbox ALBERT [20]. These experiments fully demonstrate the enhanced performance of our local adaptive strategy with respect to the standard global adaptive procedure in increasing the accuracy of the finite element solution in a subdomain.

2 Setting and Assumptions

Throughout this paper, $\Omega \subset \mathbb{R}^2$ is a bounded polygonal domain. Let $H^m(\Omega)$ be the usual Sobolev space with the norm $\|\cdot\|_{m,\Omega}$ and seminorm $|\cdot|_{m,\Omega}$, and let $\mathring{H}^m(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. Let H^{-m} be the dual of $\mathring{H}^m(\Omega)$. See [1] for details.

2.1 Discretization

Let u be the weak solution of (1.1), namely,

$$u \in \mathring{H}^1(\Omega): \qquad a(u,v) = \langle f, v\rangle, \quad \forall v \in \mathring{H}^1(\Omega), \tag{2.1}$$

where $a(u,v) = \int_\Omega A(x)\nabla u \cdot \nabla v \, dx$, and $\langle f, v\rangle = \int_\Omega f(x)v(x)dx$.

Let $\mathcal{T}_h = \{T\}$ be a family of shape-regular triangulations of Ω, which is conforming (two elements in $\mathcal{T}_h$ share at most a vertex or an side). Hereafter, h denotes the piecewise constant mesh-size of $\mathcal{T}_h$, namely $h|_T = h_T$ -the diameter of T- for all $T \in \mathcal{T}_h$; $h_\Omega = \max_{T\in\mathcal{T}} h_T$ stands for the largest element size in Ω. Let $\mathcal{S}_h$ be the set of interelement boundaries of sides S, whose diameter is h_S. We associate with $\mathcal{T}_h$, finite element spaces $S_h(\Omega) \subset H^1(\Omega)$ and $\mathring{S}_h(\Omega) = S_h(\Omega) \cap \mathring{H}^1(\Omega)$. The finite element approximation u_h of u satisfies:

$$u_h \in \mathring{S}_h(\Omega): \qquad a(u_h, v) = \langle f, v\rangle, \quad \forall v \in \mathring{S}_h(\Omega). \tag{2.2}$$

For a subdomain G of Ω, let $S_h(G)$ be the restriction of $S_h(\Omega)$ to G, and set

$$\mathring{S}_h(G) := \{v \in S_h(\Omega) : \mathrm{supp} v \subset\subset G\}.$$

Let $\mathcal{T}_h(G)$ be the set of elements intersecting G and let $\mathcal{S}_h(G)$ be set of sides S of elements $T \in \mathcal{T}_h(G)$. i.e.

$$\mathcal{T}_h(G) := \{T \in \mathcal{T}_h : T \cap G \neq 0\}, \quad \mathcal{S}_h(G) := \{S \in \mathcal{S}_h : S \subset \partial T, T \in \mathcal{T}_h(G)\}.$$

Given subdomains Ω_0 and Ω_1 as in (1.2), namely $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$, we assume that h is sufficiently fine within Ω_0 for $\Omega_h := \cup\{T \in \mathcal{T}_h(\Omega_0)\}$ to satisfy $\Omega_h \subset\subset \Omega_1$. Let d be

$$d := \operatorname{dist}(\partial\Omega_0 \backslash \partial\Omega, \partial\Omega_1 \backslash \partial\Omega) > 0. \tag{2.3}$$

2.2 Weight Function

Let V_i be the vertices of Ω and let Γ_i be the open straight edges of $\partial\Omega$ connecting V_i and V_{i+1} for $1 \le i \le M$. Let ω_i be the interior angle of Ω at V_i. Let

$$r_i(x) := |x - V_i|, \quad 1 \le i \le M,$$

be the Euclidean distance between x and vertex V_i of Ω.

Let $N(A)$ be the union of all elements of $\mathcal{T}_h$ adjacent to the close set A, namely,

$$N(A) := \bigcup \{T' \in \mathcal{T}_h : \quad A \cap T' \neq \emptyset\}. \tag{2.4}$$

Since $\mathcal{T}_h$ is shape-regular, there exists a constant $C > 0$ solely depending on the minimum angle of $\mathcal{T}_h$ such that

$$\operatorname{dist}(T, \Omega \backslash N(T)) \ge C h_T, \quad \forall T \in \mathcal{T}_h,$$

whence, for elements T not containing corner V_i,

$$r_i(x) \ge \operatorname{dist}(T, V_i) \ge \operatorname{dist}(T, \Omega \backslash N(T)) \ge C h_T, \quad \forall x \in T. \tag{2.5}$$

Let $\rho_i : \Omega \to \mathbb{R}^+$ be the following modified distance function to vertex V_i,

$$\rho_i(x) := \sqrt{r_i(x)^2 + h^2(x)}, \qquad \forall x \in \Omega.$$

By virtue of (2.5), we readily deduce

$$r_i(x) < \rho_i(x) \le C r_i(x) \quad \forall x \in T. \tag{2.6}$$

provided $V_i \notin T$, as well as *non-oscillation* property

$$\max_{x \in N(T)} \rho_i(x) \le \Lambda \min_{x \in N(T)} \rho_i(x), \quad \forall T \in \mathcal{T}_h, \tag{2.7}$$

where Λ solely depends on C of (2.6). In addition to (2.5), (2.7) hinges on the Lipschitz regularity of r_i in case $V_i \notin T$, or otherwise on the equivalence of r_i and h near V_i.

Given a M-tuple $\alpha = (\alpha_i)_{i=1}^M$, we define the mesh-dependent *weight function*

$$\sigma_\alpha(x) := \min_{1 \le i \le M} \rho_i^{\alpha_i}(x), \tag{2.8}$$

which also satisfies (2.7). We note that for $0 \le \alpha_i < 1$, we have the crucial fact

$$h(x)\sigma_{-\alpha}(x) \le \min_{1 \le i \le M} h(x)^{1-\alpha_i} \le h_\Omega^{1-|\alpha|_\infty}, \quad \forall x \in \Omega. \tag{2.9}$$

2.3 Basic Assumptions

We make the following basic assumptions (A.1) - (A.3) concerning the domain Ω and the finite element space $S_h(\Omega)$.

A.1 Regularity. Let $\omega_1, \cdots, \omega_I$ be the internal angles of the reentrant corners of Ω; thus $\pi < \omega_i < 2\pi$ if $1 \le i \le I$ and $\omega_i < \pi$ if $I < i \le M$. Let

$$\beta_i := \max\Big(0, 1 - \frac{\pi}{\omega_i}\Big), \quad \forall 1 \le i \le M, \tag{2.10}$$

and let $\beta := (\beta_i)_{i=1}^M$ be the associate M-tuple. We assume that the solution $u \in \mathring{H}^1(\Omega)$ of (1.1) admits an expansion in terms of a regular function w and singular functions ζ_i as follows

$$u = w + \sum_{i=1}^{I} \kappa_i \zeta_i, \tag{2.11}$$

with $w \in H^2(\Omega) \cap \mathring{H}^1(\Omega)$ and the coefficients (intensity factors) κ_i satisfying

$$\|w\|_{H^2(\Omega)} + \sum_{i=1}^{I} |\kappa_i| \le C\|f\|_{L^2(\Omega)}. \tag{2.12}$$

This property is satisfied by the scalar equation (1.1) over polygonal domains in 2d [9, 13, 14, 15]. The singular functions ζ_i are defined as follows. Let polar coordinates (r_i, θ_i) be chosen at the vertex V_i so that the interior angle ω_i is spanned by the two half lines $\theta_i = 0$ and $\theta_i = \omega_i$. The singular function ζ_i can then be written as

$$\zeta_i(r_i, \theta_i) = \phi_i(r_i) r_i^{\pi/\omega_i} \sin\big((\pi/\omega_i)\theta_i\big), \tag{2.13}$$

where $\phi_i(r_i)$ are smooth cut-off functions which equal 1 in a neighborhood of $r_i = 0$ and possess a non-overlapping support. Thus the solution around the corner V_i behaves like $r_i^{\gamma_i}$ with $\gamma_i = \pi/\omega_i$.

Let h_i be the meshsize at the corner V_i, let $B(x, r)$ be a close ball of center x and radius r, and set $\Omega_i := \Omega \backslash B(V_i, h_i)$. Since $\gamma_i + \beta_i = 1$ for $1 \le i \le I$, with β_i as in (2.10), we readily conclude that

$$\|\sigma_\beta D^2 \zeta_i\|_{0,\Omega_i} \le C|\log h_i|^{1/2}. \tag{2.14}$$

A.2. Local Approximability. There exists an integer $k \ge 1$ (polynomial degree) and an interpolation operator $I_h : L^2(\Omega) \to \mathring{S}_h(\Omega)$ such that for all $v \in H^m(\Omega)$ and $T \in \mathcal{T}$

$$\|v - I_h v\|_{j,T} \le C h_T^{m-j} \|D^m v\|_{0,N(T)}, \quad 0 \le j \le m \le k+1, \tag{2.15}$$

where $N(T)$ is defined in (2.4). In addition, if $\operatorname{supp} v \subset \Omega_0$, then $I_h v \in S_0^h(\Omega_1)$. In view of (2.7) for σ_α, we infer the local weighted interpolation estimate

$$\|D^j(v - I_h v)\sigma_\alpha\|_{0,T} \le C h_T^{m-j} \|\sigma_\alpha D^m v\|_{0,N(T)}, \quad j = 0, 1 \le m \le 2. \tag{2.16}$$

A.3. Locally smooth boundary. Let $D_0 \subset\subset D_1$ be such that $\mathrm{diam}(D_0) = 1$ and $\mathrm{diam}(D_1) = 2$, there exists D_0' with $D_0 \subset\subset D_0' \subset\subset D_1$, and an integer $t \geq 0$ such that for any $v \in C_0^\infty(D_0')$, the solution of the auxiliary problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla V) = v, & \text{in } D_0', \\ V = 0, & \text{on } \partial D_0', \end{cases} \tag{2.17}$$

satisfies

$$\|V\|_{j+2,D_0'} \leq C\|v\|_{j,D_0'}, \quad 0 \leq j \leq t. \tag{2.18}$$

A.3'. Globally smooth boundary. There exists an integer $t \geq 0$ such that for any $f \in C_0^\infty(\Omega)$, the solution of (1.1) satisfies

$$\|u\|_{j+2,\Omega} \leq C\|f\|_{j,\Omega}, \quad 0 \leq j \leq t. \tag{2.19}$$

2.4 Error and Residual

Setting $e_h := u - u_h$ and integrating by parts $a(e_h, \phi)$ yields the following error-residual relation

$$a(e_h, \phi) = \sum_{T \in \mathcal{T}_h} \int_T R_T \phi + \sum_{S \in \mathcal{S}_h} \int_S J_S \phi, \quad \forall \phi \in \mathring{H}^1(\Omega). \tag{2.20}$$

where R_T is the interior residual and J_S is the jump residual, which are defined as follows:

$$R_T := \big(f + \operatorname{div}(A\nabla u_h)\big)|_T, \quad \forall T \in \mathcal{T}_h; \qquad J_S := [A\nabla u_h \cdot \nu_S], \quad \forall S \in \mathcal{S}_h.$$

Subtracting (2.2) from (2.1), we get the *Galerkin orthogonality* equation

$$a(e_h, \phi_h) = 0 \quad \forall \phi_h \in \mathring{S}_h(\Omega). \tag{2.21}$$

This, in conjunction with (2.20), leads to the final error-residual relation:

$$a(e_h, \phi) = \sum_{T \in \mathcal{T}_h} \int_T R_T(\phi - I_h\phi) + \sum_{S \in \mathcal{S}_h} \int_S J_S(\phi - I_h\phi), \quad \forall \phi \in \mathring{H}^1(\Omega). \tag{2.22}$$

2.5 Weighted Residual Estimators

For any given integer $j \leq 1$ we introduce the local error indicator

$$\eta_j^2(T) := h_T^{4-2j}\|R_T\|_{0,T}^2 + h_T^{3-2j}\|J_T\|_{0,\partial T}^2, \quad \forall T \in \mathcal{T}_h \tag{2.23}$$

where J_T stands for the flux jumps across ∂T. If D is a subdomain of Ω, then the H^j-type error indicator over D is given by

$$\eta_j^2(D) := \sum_{T \in \mathcal{T}_h(D)} \eta_j^2(T).$$

It is well known that $\eta_1(\Omega)$ provides a global upper bound for the error e_h in $H^1(\Omega)$ [2, 3, 11, 22], and $\eta_0(\Omega)$ does so for the $L^2(\Omega)$ norm provided Ω is convex (or smooth) [11]. The latter results from a standard duality argument. We have

$$\eta_{j-k}(D) \leq h_D^k \eta_j(D) \quad \forall k \geq 1. \tag{2.24}$$

We now introduce the novel *weighted L^2-type* local error indicators

$$\eta_{0,-\beta}^2(T) := h_T^4 \|R_T \sigma_{-\beta}\|_{0,T}^2 + h_T^3 \|J_T \sigma_{-\beta}\|_{0,\partial T}^2, \tag{2.25}$$

where β satisfies (2.10) In contrast to (2.23) with $j=0$, note the occurrence of the weight $\sigma_{-\beta}$. We will see in Section 3.2 that this modification is responsible for the upper bound of the L^2 error for general polygonal domains Ω, even with reentrant corners. In view of (2.9) we have the key relation

$$\eta_{0,-\beta}(T) \leq h_T^{1-|\beta|_\infty} \eta_1(T) \quad \forall\, T \in \mathcal{T}_h. \tag{2.26}$$

The corresponding weighted error indicator for a region $D \subset \Omega$ is

$$\eta_{0,-\beta}^2(D) := \sum_{T \in \mathcal{T}_h(D)} \eta_{0,-\beta}^2(T).$$

Owing to (2.26), $\eta_{0,-\beta}(D)$ is asymptotically infinitesimal with respect to $\eta_1(D)$

$$\eta_{0,-\beta}(D) \leq h_D^{1-|\beta|_\infty} \eta_1(D). \tag{2.27}$$

3 Local a Posteriori Error Estimators

We derive computable upper bounds for the error e_h in $H^1(\Omega_0)$, which account for the pollution effect from $\Omega \backslash \Omega_0$. In section 3.1 we obtain such estimates in terms of the non-computable quantity $\|u - u_h\|_{0,\Omega}$, as in [24]. We further estimate this L^2 norm a posteriori in section 3.2, and assemble altogether in section 3.3 and section 3.4.

3.1 Localization of H^1-Error

To localize the estimate to the subdomain Ω_0, we introduce a cut-off function $\omega \in C^\infty(\Omega_1)$ with $\omega = 1$ in Ω_0 and $\omega = 0$ in $\Omega \backslash \Omega_1$, such that for $l \geq 0$

$$\|\omega\|_{l,\infty,\Omega_1} \leq \Lambda d^{-l}, \tag{3.1}$$

where d is as in (2.3). Since $\omega e_h \in \mathring{H}^1(\Omega_1)$, we have

$$\|e_h\|_{1,\Omega_0}^2 \leq \|\omega e_h\|_{1,\Omega_1}^2 \leq C \|\nabla(\omega e_h)\|_{0,\Omega_1}^2. \tag{3.2}$$

Lemma 3.1. *Let (A.2) be valid. Then there exist constants C_1 and C_2, only dependent on shape-regularity of $\mathcal{T}_h$, such that*

$$\|e_h\|_{1,\Omega_0} \leq C_1 \eta_1(\Omega_1) + \frac{C_2}{d} \|e_h\|_{0,\Omega_1 \backslash \Omega_0}. \tag{3.3}$$

Remark 3.2. In Lemma 3.1, the domains Ω_0 and Ω_1 are arbitrary but fixed, d displays the precise dependence of constants on domain size, and d needs to be adequately large for $\Omega_1 \backslash \Omega_0$ to contain several layers of elements.

Proof of Lemma 3.1. The right-hand side of (3.2) can be estimated as follows:

$$\begin{aligned} C\|\nabla(\omega e_h)\|^2_{L^2(\Omega_1)} &\leq \int_\Omega A\nabla(\omega e_h)\cdot\nabla(\omega e_h) \\ &= \int_\Omega A\nabla e_h\cdot\nabla(\omega^2 e_h) + \int_\Omega e_h^2(A\nabla\omega)\cdot\nabla\omega. \end{aligned} \tag{3.4}$$

Since $\phi = \omega^2 e_h \in \mathring{H}^1(\Omega)$, it follows from (2.22) that

$$\int_\Omega A\nabla e_h\cdot\nabla(\omega^2 e_h) = \sum_{T\in\mathcal{T}_h(\Omega_1)}\int_T R_T(\phi - I_h\phi) + \sum_{S\in\mathcal{S}_h(\Omega_1)}\int_S J_S(\phi - I_h\phi).$$

Using (2.15) with $j = 0, 1$, together with the trace inequality for $v = \phi - I_h\phi$

$$\|v\|^2_{0,\partial T} \leq Ch_T^{-1}\|v\|^2_{0,T} + Ch_T\|\nabla v\|^2_{0,T}, \tag{3.5}$$

we infer that

$$\begin{aligned} \int_\Omega A\nabla e_h\cdot\nabla(\omega^2 e_h) &\leq C\eta_1(\Omega_1)\|\nabla\phi\|_{0,\Omega} \\ &\leq C\eta_1^2(\Omega_1) + \frac{1}{4}\|\nabla(\omega e_h)\|^2_{0,\Omega} + \frac{C}{d^2}\|e_h\|^2_{0,\Omega_1\backslash\Omega_0}. \end{aligned} \tag{3.6}$$

On the other hand, the Cauchy-Schwarz inequality, in conjunction with (3.1), yields

$$\int_\Omega e_h^2 A\nabla\omega\cdot\nabla\omega \leq \frac{C}{d^2}\|e_h\|^2_{0,\Omega_1\backslash\Omega_0}. \tag{3.7}$$

Combining (3.6) and (3.7), we prove the asserted estimate. □

The local error indicator $\eta_1(\Omega_1)$ measures the H^1 interpolation error in the slightly larger domain Ω_1 a posteriori. Note that $\eta_1(\Omega_1)$ need not necessarily provide an upper bound for the true local error $\|e_h\|_{1,\Omega_0}$, as indicated in (3.3), because it does not measure any possible pollution effect arising from outside of Ω_1. The pollution effect is hidden in the non-computable quantity $\|e_h\|_{0,\Omega_1\backslash\Omega_0}$. We now derive an a posteriori upper bound for this term.

3.2 Global Weighted L^2-Error Estimators

We now show that $\eta^2_{0,-\beta}(\Omega)$ gives an upper bound for the global L^2 error in arbitrary polygons provided β is suitably chosen.

Lemma 3.3. *Let (A.1) and (A.2) be valid. Let β be the M-tuple of (A.1) and h_i be the meshsize at the corner V_i of Ω. Then there exists a constant C_3, solely dependent on shape-regularity of $\mathcal{T}_h$, such that*

$$\|e_h\|_{0,\Omega} \leq C_3 \max_{1\leq i\leq I} |\log h_i|^{1/2}\ \eta_{0,-\beta}(\Omega). \tag{3.8}$$

Proof. We use a duality argument. For $\varphi \in C_0^\infty(\Omega)$, let ψ be the solution of

$$\begin{cases} -\operatorname{div}(A\nabla\psi) = \varphi, & \text{in } \Omega, \\ \qquad\qquad \psi = 0, & \text{on } \partial\Omega. \end{cases} \tag{3.9}$$

From the expansion formula (2.11), ψ can be expressed as

$$\psi = \sum_{i=1}^{I} \kappa_i \zeta_i + w, \tag{3.10}$$

with $w \in \mathring{H}^1(\Omega) \cap H^2(\Omega)$ and coefficients κ_i satisfying (2.12) with f replaced by φ. In view of (2.22) and (3.9), we deduce

$$\langle e_h, \varphi\rangle = a(e_h, \psi) = \sum_{T\in\mathcal{T}_h} \int_T R_T(\psi - I_h\psi) + \sum_{S\in\mathcal{S}_h} \int_S J_S(\psi - I_h\psi). \tag{3.11}$$

We now have to estimate the error $\psi - I_h\psi$. To this end, we examine the regular function w and the singular functions ζ_i separately.

For w we simply apply (2.12) and (2.15), coupled with (3.5), to obtain the standard L^2-estimate

$$a(e_h, w) = C\eta_0(\Omega)\|D^2 w\|_{0,\Omega} \leq C\eta_{0,-\beta}(\Omega)\|\varphi\|_{0,\Omega}. \tag{3.12}$$

Note that we have used the property $\eta_0(\Omega) \leq \eta_{0,-\beta}(\Omega)$. To deal with ζ_i we must distinguish whether or not $N(T)$ (or $N(S)$) touches V_i. If it does not, we then utilize (2.16) with $j = 0$ to arrive at

$$\begin{aligned} \int_T R_T(\zeta_i - I_h\zeta_i) &\leq \|\sigma_{-\beta} R_T\|_{0,T} \|\sigma_\beta(\zeta_i - I_h\zeta_i)\|_{0,T} \\ &\leq C\|h^2\sigma_{-\beta}R_T\|_{0,T}\|\sigma_\beta D^2\zeta_i\|_{0,N(T)}. \end{aligned}$$

In case $V_i \in N(T)$, the function ζ_i is singular in $N(T)$ but it satisfies the following crucial property:

$$\int_{B(V_i,Ch_i)} \sigma_\beta^2 \zeta_i^2 \leq Ch_i^{2\beta_i} \int_0^{Ch_i} r^{2\gamma_i+1} dr \leq Ch_i^4. \tag{3.13}$$

We thereby make use of (2.16) with $j = m = 0$ to get

$$\int_T R_T(\zeta_i - I_h\zeta_i) \leq C\|\sigma_{-\beta}R_T\|_{0,T}\|\sigma_\beta\zeta_i\|_{0,N(T)} \leq C\|h^2\sigma_{-\beta}R_T\|_{0,T}.$$

Adding over all $T \in \mathcal{T}_h$ and invoking (2.14), we realize that

$$\sum_{T\in\mathcal{T}_h} \int_T R_T(\zeta_i - I_h\zeta_i) \le C\Big(1 + \|\sigma_\beta D^2\zeta_i\|_{0,\Omega_i}\Big)\Big(\sum_{T\in\mathcal{T}_h} \|h^2\sigma_{-\beta}R_T\|_{0,T}^2\Big)^{1/2}$$
$$\le C|\log h_i|^{1/2}\ \eta_{0,-\beta}(\Omega).$$

We argue similarly with $\int_S J_S(\zeta_i - I_h\zeta_i)$, this time exploiting (2.16) for both $j = 0, 1$ along with (3.5) and the following substitute for (3.13)

$$h_i \int_{B(V_i,Ch_i)} \sigma_\beta^2 |\nabla\zeta_i|^2 \le Ch_i^{1+2\beta_i} \int_0^{Ch_i} r^{2\gamma_i - 1} dr \le Ch_i^3.$$

Together with the preceding estimate, this leads to

$$a(e_h, \zeta_i) \le C|\log h_i|^{1/2}\ \eta_{0,-\beta}(\Omega).$$

By virtue of (2.12) and (3.12), we finally discover

$$a(e_h, \psi) \le C \max_{1\le i\le I} |\log h_i|^{1/2}\ \eta_{0,-\beta}(\Omega)\|\varphi\|_{0,\Omega},$$

which implies the asserted estimate. $\square$

Remark 3.4. In light of (2.11) and (2.13) we obtain the usual *global* regularity result $\psi \in H^{1+s}(\Omega)$ with $s < \pi/\bar\omega$ with $\bar\omega = \max(\omega_1, \omega_2, \cdots, \omega_I)$. Employing this result in the duality argument of Lemma 3.3 we would end up with the *suboptimal* L^2 estimate

$$\|e_h\|_{0,\Omega} \le C\eta_{1-s}(\Omega).$$

Since $\beta_i = 1 - \gamma_i < 1 - s$ for $1 \le i \le I$, we easily see that $\sigma_{-\beta} \le h^{s-1}$, whence

$$\eta_{0,-\beta}(\Omega) \le \eta_{1-s}(\Omega).$$

Remark 3.5. Our result (3.8) resembles somehow those of Becker and Rannacher [6, 7], the difference being that they advocate the numerical solution of the dual problem (3.9) while we examine its influence analytically. However in both cases the estimators contain weights. If the goal is to evaluate numerically a linear functional of u_h, then an explicit expression for φ may be known in advance thereby making the approach of [6, 7] rigorous. But in evaluating the L^2 norm of the error e_h in an *arbitrary polygon* we have to maximize over φ and thus the choice of φ becomes critical, an important issue completely circumvented here.

3.3 Local H^1-Error Estimators: Rough Boundary

We now derive several results upon conbining Lemma 3.1 and Lemma 3.3.

Theorem 3.6. *Let (A.1) and (A.2) be valid. Let β be the M-tuple of (A.1) and h_i the meshsize at the corner V_i of Ω for $1 \le i \le I$, and let them satisfy*

$$h_{\Omega_1}^{1-|\beta|_\infty} \max_{1\le i\le I} |\log h_i|^{1/2} \le 1.$$

Then there exist constants C_1 and C_2, solely dependent on the shape-regularity of $\mathcal{T}_h$, such that

$$\|e_h\|_{1,\Omega_0} \le C_1 \eta_1(\Omega_1) + \frac{C_2}{d} \max_{1\le i\le I} |\log h_i|^{1/2} \ \eta_{0,-\beta}(\Omega\backslash\Omega_1). \tag{3.14}$$

Proof. Lemma 3.1 and Lemma 3.3 yield

$$\|e_h\|_{1,\Omega_0} \le C_1 \eta_1(\Omega_1) + \frac{C_2}{d} \max_{1\le i\le I} |\log h_i|^{1/2} \ \big(\eta_{0,-\beta}(\Omega_1) + \eta_{0,-\beta}(\Omega\backslash\Omega_1)\big).$$

The assertion follows trivially from (2.27), namely $\eta_{0,-\beta}(\Omega_1) \le h_{\Omega_1}^{1-|\beta|_\infty} \eta_1(\Omega_1)$. □

Remark 3.7. Since $\|e_h\|_{1,\Omega_0} \le \|e_h\|_{1,\Omega}$, we obviously have

$$\|e_h\|_{1,\Omega_0} \le C\big(\eta_1(\Omega_1) + \eta_1(\Omega\backslash\Omega_1)\big).$$

Even though this is a global upper bound, according to (2.26) and (2.27), it is much too crude to control the local error in $H^1(\Omega_0)$. We could thus expect that (3.14) would give rise to a more efficient algorithm for adaptive mesh refinement with error control in Ω_0. We quantify this gain in Remarks 3.8 and 3.9.

Remark 3.8. Consider a model domain Ω with only a reentrant corner at the origin, such as the L-shaped domain. Moreover, suppose that Ω_0 contains the origin, and T is an element in the complement of Ω_1. Hence

$$\eta_{0,-\beta}(T) \le \frac{h_T}{\mathrm{dist}(\partial\Omega_1, 0)^\beta} \eta_1(T), \tag{3.15}$$

which shows a gain of a full power of meshsize with respect to the energy estimator. This improvement is responsible for the superior performance of our algorithm in Example 4.2.

Remark 3.9. Suppose now that in Remark 3.8 the corner does not belong to Ω_0. In this case, we have

$$\eta_{0,-\beta}(T) \le \frac{h_T}{\max(h_T, r_T)^\beta} \eta_1(T), \tag{3.16}$$

where $r_T = \mathrm{dist}(T, 0)$. We thus realize that there is a gain of a fractional power $h_T^{1-\beta}$ of meshsize near the origin and a full power away from the origin. This improvement explains the enhanced performance of Example 4.3.

Remark 3.10. The following *suboptimal* estimate is derived in [24]

$$\|e_h\|_{1,\Omega_0} \leq C_1\eta_1(\Omega_1) + C_2\eta_{1-s}(\Omega\backslash\Omega_1), \tag{3.17}$$

where s is as in Remark 3.4. In [24] the second term on the right hand side of (3.17) is claimed to be of higher order than the first term and therefore negligible. Since the estimators $\eta_1(\Omega_1)$ and $\eta_{1-s}(\Omega\backslash\Omega_1)$ are defined in complementary sets and account for different effects, such an assertion is rarely true, and is certainly false in the early stages of an adaptive procedure. Our estimator (3.14), instead, replaces $\eta_{1-s}(\Omega\backslash\Omega_1)$ with the *computable* weighted estimator $\eta_{0,-\beta}(\Omega\backslash\Omega_1)$, which turns out to be sharper and thereby gives rise a coarser meshes; see Remark 3.4.

3.4 Local H^1-Error Estimators: Smooth Boundary

In this section we prove that if $\partial\Omega$ is sufficiently smooth in the sense of (A.3), then the local error $\|e_h\|_{0,\Omega_1\backslash\Omega_0}$ in Lemma 3.1 can be replaced by a weaker *negative* norm; this entails higher powers of the meshsize. Recall that the negative norm for a given function u is defined as

$$\|u\|_{-m,\Omega} = \sup_{\substack{v\in C_0^\infty(\Omega)\\ |v|_{m,\Omega}=1}} \langle u, v\rangle.$$

Lemma 3.11. *Let (A.2) and (A.3) be valid, D_0, D_1 be as in (A.3), $0 \leq j \leq \min(t, k-1)$ with k as in (A.2). There exist constants C_1 and C_2, solely dependent on the shape-regularity of $\mathcal{T}_h$, such that*

$$\|e_h\|_{-j+1,D_0} \leq C_1\eta_{-j+1}(D_1) + C_2\|e_h\|_{-j,D_1\backslash D_0}. \tag{3.18}$$

Proof. We use duality argument. Let D_0' be as in (A.3), $\omega \in C_0^\infty(D_0')$ satisfy (3.1) with $d = 1$. For any $\varphi \in C_0^\infty(D_0)$, let ψ satisfy (3.9) with Ω replaced by D_0'. Then

$$\begin{aligned}\langle e_h, \varphi\rangle &= \langle \omega e_h, \varphi\rangle = a(\omega e_h, \psi)\\ &= a(e_h, \omega\psi) + \int_{D_0'} \big(\operatorname{div}(A(x)\psi\nabla\omega) + A(x)\nabla\psi\nabla\omega\big)e_h.\end{aligned} \tag{3.19}$$

Consequently by (2.22) and Assumption (A.2), we obtain

$$\begin{aligned}|\langle e_h, \varphi\rangle| &\leq C\eta_{-j+1}(D_0')\|D^{j+1}(\omega\psi)\|_{0,D_0'}\\ &\quad + C\|e_h\|_{-j,D_1\backslash D_0}|\operatorname{div}(A(x)\psi\nabla\omega) + A(x)\nabla\psi\nabla\omega|_{j,D_1\backslash D_0}.\end{aligned} \tag{3.20}$$

We now resort to Leibniz formula, (3.1) with $d = 1$, Assumption (A.3) and Poincaré inequality to get

$$\|D^{j+1}(\omega\psi)\|_{0,D_0'} \leq C\|\varphi\|_{j-1,D_0} \leq C|\varphi|_{j-1,D_0}.$$

The estimate for the second term on the right hand side of (3.20) can be handled similarly. We therefore conclude the proof. ∎

With the help of Lemma 3.11 and scaling argument, we can now prove

Lemma 3.12. *Let Ω_0 and Ω_1 satisfy (2.3), and let Assumptions (A.2) and (A.3) be valid. If $h_{\Omega_0}/d < 1$, then there exists constants C_1 and C_2, solely dependent on the shape-regularity of $\mathcal{T}_h$ and $0 \le j \le \min(t, k-1)$, such that*

$$\|e_h\|_{0,\Omega_0} \le C_1 \eta_0(\Omega_1) + \frac{C_2}{d^j} \|e_h\|_{-j,\Omega_1}. \tag{3.21}$$

Proof. We follow an argument of Schatz and Wahlbin [18, 19], and we sketch it below for completeness.
1. Let $B_d := B(x_0, d)$ be the ball of center $x_0 \in \Omega_0$ and radius d. We claim that

$$\|e_h\|_{-j+1,B_d} \le C_1 \eta_{-j+1}(B_{2d}) + \frac{C_2}{d} \|e_h\|_{-j,B_{2d}\setminus B_d}, \tag{3.22}$$

where constants C_1 and C_2 depend only on mesh shape-regularity and j, but are otherwise independent of B_d and B_{2d}.

We begin the proof of (3.22) by introducing a new variable $y := x/d$. For any given function ξ, denote $\widetilde{\xi}(y) := \xi(yd)$; for any given set D, denote $\widetilde{D} := D/d$. Then, (1.1) becomes

$$\begin{cases} -\operatorname{div}_y(\widetilde{A}(y)\nabla\widetilde{u}(y)) = d^2 \widetilde{f}(y), & \text{in } \widetilde{\Omega}, \\ \widetilde{u}(y) = 0, & \text{on } \partial\widetilde{\Omega}. \end{cases} \tag{3.23}$$

Since (3.23) has the same modulus of ellipticity as (1.1), applying Lemma 3.11 to the error $\widetilde{e}_{\tilde{h}}(y) := e_h(dy)$ with $D_0 = B_1$ and $D_1 = B_2$, we obtain

$$\|\widetilde{e}_{\tilde{h}}\|_{-j+1,B_1} \le C_1 \widetilde{\eta}_{-j+1}(B_2) + C_2 \|\widetilde{e}_{\tilde{h}}\|_{-j,B_2\setminus B_1}. \tag{3.24}$$

Transforming back to the original variable x, we have

$$\|\widetilde{e}_{\tilde{h}}\|_{-j+1,B_1} = d^{-j}\|e_h\|_{-j+1,B_d}, \quad \|\widetilde{e}_{\tilde{h}}\|_{-j,B_2\setminus B_1} = d^{-j-1}\|e_h\|_{-j,B_{2d}\setminus B_d},$$
$$\widetilde{\eta}_{-j+1}(B_2) = d^{-j}\eta_{-j+1}(B_{2d}).$$

These equalities, combined with (3.24), lead to (3.22).
2. Applying (3.22) with $j = 1$, we get

$$\|e_h\|_{0,B_d} \le C_1 \eta_0(B_{2d}) + \frac{C_2}{d} \|e_h\|_{-1,B_{2d}\setminus B_d}.$$

We now reapply (3.22) to estimate $\|e_h\|_{-1,B_{2d}}$. Since $h_{B_{2^i d}}/d < 1$ for $1 \le i \le j-1$, we see that $d^{-1}\eta_{-i}(B_{2^{i+1}d}) \le \eta_{-i+1}(B_{2^{i+1}d})$. Iterating (3.22), we get

$$\|e_h\|_{0,B_d} \le C_1 \eta_0(B_{2^j d}) + \frac{C_2}{d^j} \|e_h\|_{-j,B_{2^j d}\setminus B_{2^{j-1} d}}.$$

Equivalently, with constants C_1 and C_2 depending on j, we have

$$\|e_h\|_{0,B_{2^{-j} d}} \le C_1 \eta_0(B_d) + \frac{C_2}{d^j} \|e_h\|_{-j,B_d\setminus B_{d/2}}. \tag{3.25}$$

3. Cover Ω_0 with balls $B(x_i, 2^{-j}d)$ satisfying the finite overlapping property with $x_i \in \Omega_0$ for $1 \le i \le n$. Then (3.25) gives

$$\begin{aligned} \|e_h\|^2_{0,\Omega_0} &\le \sum_{i=1}^n \|e_h\|^2_{0,B(x_i,2^{-j}d)} \\ &\le 2C_1^2(K+1)\eta_0(\Omega_1)^2 + \frac{2C_2^2}{d^{2j}} \sum_{i=1}^n \|e_h\|^2_{-j,B(x_i,d)}, \end{aligned} \tag{3.26}$$

where K is an upper bound, independent of n, for the number of adjacent balls intersecting $B(x_i, d), 1 \le i \le n$. The set $\{B(x_i, d)\}_{i=1}^n$ can be split into $\{\mathcal{G}_k\}_{k=1}^K$ where each $\mathcal{G}_k$ is a collection of disjoint balls $B(x_i, d)$ and no ball belongs to more than one $\mathcal{G}_k$.

Lemma 1.1 in [18] asserts that if D_i are disjoint domains for $1 \le i \le I$ and $D = \cup_{i=1}^I D_i$, then $\sum_{i=1}^I \|e_h\|^2_{-j,D_i} \le \|e_h\|^2_{-j,D}$. This yields

$$\sum_{i=1}^n \|e_h\|^2_{-j,B(x_i,d)} = \sum_{k=1}^K \sum_{B(x_i,d)\in\mathcal{G}_k} \|e_h\|^2_{-j,B(x_i,d)} \le K\|e_h\|^2_{-j,\Omega_1}. \tag{3.27}$$

Combining (3.26) and (3.27), we finally conclude the proof of (3.21). □

As in the proof of Lemma 3.11, a duality argument will lead to the following a posteriori estimators for the error in negative norm.

Corollary 3.13. *Let (A.2) and (A.3') be valid. There exists a constant C, solely dependent on shape-regularity of $\mathcal{T}_h$, such that*

$$\|e_h\|_{-j,\Omega} \le C\eta_{-j}(\Omega). \tag{3.28}$$

We can now improve estimate (3.14) provided that the domain is smooth.

Theorem 3.14. *Let (A.2) and (A.3') be valid and $h_{\Omega_0}/d < 1$. There exist constants C_1, C_2, C_3 and C_4, solely dependent on shape-regularity of $\mathcal{T}_h$ and the positive number j, such that for $0 \le j \le \min(t, k-1)$ we have*

$$\|e_h\|_{1,\Omega_0} \quad\le\quad C_1\eta_1(\Omega_1) + \frac{C_2}{d^{j+1}}\eta_{-j}(\Omega\backslash\Omega_1). \tag{3.29}$$

If $k \ge 2$ and $t \ge 1$, then

$$\|e_h\|_{0,\Omega_0} \le C_3\eta_0(\Omega_0) + \frac{C_4}{d^j}\eta_{-j}(\Omega\backslash\Omega_1). \tag{3.30}$$

Proof. Let $\widetilde{\Omega}_1$ be such that $\Omega_0 \subset\subset \widetilde{\Omega}_1 \subset\subset \Omega_1$. It follows from Lemma 3.1 that

$$\|e_h\|_{1,\Omega_0} \le C_1\eta_1(\widetilde{\Omega}_1) + \frac{C_2}{d}\|e_h\|_{0,\widetilde{\Omega}_1\backslash\Omega_0}.$$

For the second term in the right hand side of above inequality, we apply Lemma 3.12 with Ω_0 replaced by $\widetilde{\Omega}_1$, and note that $\eta_0(\Omega_1)/d < \eta_1(\Omega_1)$, we obtain

$$\|e_h\|_{1,\Omega_0} \leq C_1\eta_1(\Omega_1) + \frac{C_2}{d^{j+1}}\|e_h\|_{-j,\Omega_1}. \tag{3.31}$$

(3.29) is proved by substituting the last term of (3.31) with $\|e_h\|_{-j,\Omega}$ and applying Corollary 3.13. (3.30) is a result of Lemma 3.12 combined with Corollary 3.13. □

Remark 3.15. Theorem 3.14 provides an a posteriori version of an important result for higher order elements: they behave more locally since for any $T \subset \Omega\backslash\Omega_1$,

$$\eta_{-j}(T) \leq \left(\frac{h_T}{d}\right)^{j+1}\eta_1(T), \quad \forall\, j \geq 0.$$

There is a gain of power h_T^{j+1} with respect to the energy estimator. This improvement is the primary reason for the superior performance of our algorithm in Example 4.4.

Remark 3.16. For simplicity, we assume that the boundary is matched in an isoparametric fashion.

We conclude this section with an application of Theorem 3.14, namely the case of convex domains. This result is also implied by [24].

Corollary 3.17. *Let Ω be a convex polygon, Assumption (A.2) be valid, there exists constants C_1, C_2, solely dependent on the shape-regularity of $\mathcal{T}_h$, such that*

$$\|e_h\|_{1,\Omega_0} \leq C_1\eta_1(\Omega_1) + \frac{C_2}{d}\eta_0(\Omega\backslash\Omega_1). \tag{3.32}$$

Proof. It suffices to see that for convex domains (A.3') is valid with $t = 0$. □

4 Numerical Experiments

In this section we use the local a posteriori error estimators of §3 to perform automatic, self-adaptive mesh refinements. We compare the numerical results obtained with our estimators and the usual (global) estimators in the examples below.

4.1 Adaptive Refinement Strategy

To have a suitable mesh balance inside and outside the local region Ω_0, we resort to the *equidistribution* strategy, which can be justified in heuristic grounds as in [3], using calculus of variations. Such strategy consists of equalizing all element indicators, that is $\eta_1(T)$ and $\eta_{0,-\beta}(T)$, which we denote collectively with the symbol ζ_T. This is done iteratively via the marking strategy of Dörfler [10]: given a parameter $0 < \theta < 1$ mark a subset $\hat{\mathcal{T}}_h$ of $\mathcal{T}_h$ (with largest element indicators) ζ_T such that

$$\sum_{T \in \hat{\mathcal{T}}_h} \zeta_T^2 \geq \theta \sum_{T \in \mathcal{T}_h} \zeta_T^2.$$

The idea is to refine a subset of $\mathcal{T}_h$ which amounts to a fixed proportion of the total error. This strategy is shown to converge in [10, 16] for the usual global strategy. We do not know if such a result is still valid in the present local context.

The experiments were performed within the F.E. toolbox ALBERT [20], which uses bisection for refinement. In contrast to the global Guaranteed Error Reduction Strategy (GERS) of Dörfler, our new strategy is called Local Error Refinement Strategy (LERS).

4.2 Example: Singular Solution

We consider a L-shaped domain $\Omega = \{|x| < 4, |y| < 4\} \backslash \{0 < x < 4, -4 < y < 0\}$ and Possion equation with exact solution u in polar coordinates

$$u(r, \theta) = r^{2/3} \sin((2/3)\theta),$$

and a nonvanishing Dirichlet boundary condition on $\partial\Omega$. We use piecewise quadratic conforming finite element, that is $k = 2$ in (2.15).

We apply GERS of Dörfler [10] and our refinement strategy (Theorem 3.6). Let $\Omega_0 = B(x_0, 0.5) \cap \Omega, \Omega_1 = B(x_0, 1)$, here $x_0 = (0, 0)$. We compare the local norm $\|e_h\|_{1,\Omega_0}$ in using these two refinement strategies.

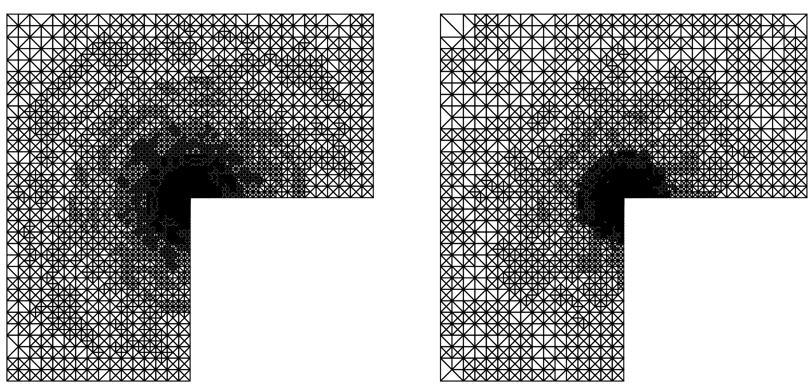

Figure 1: Example 4.2. Meshes of GERS (left) and LERS (right) with DOFs of about 24000 for $\Omega_0 = B(x_0, 0.5)$ and $x_0 = (0, 0)$; $\|e_h\|_{1,\Omega_0} \approx 2.13364540e-4$ for GERS and $\|e_h\|_{1,\Omega_0} \approx 1.69867099e-4$ for LERS.

Figure 1 displays the meshes obtained with GERS and LERS for about the same DOFs of 24000. LERS achieves a more accurate approximation in the

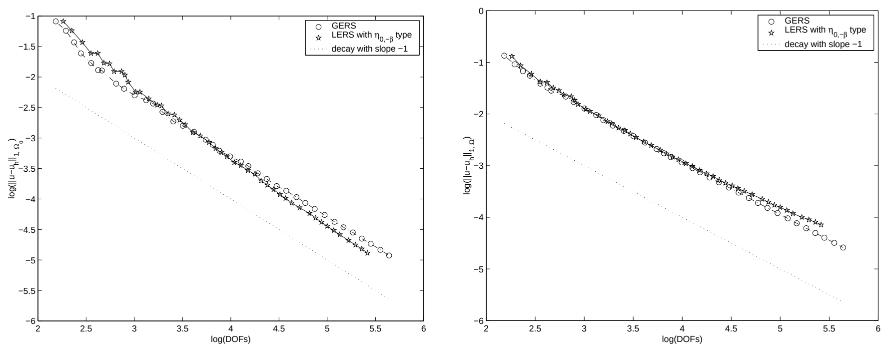

Figure 2: Example 4.2. Quasioptimality of LERS for $\|e_h\|_{1,\Omega_0}$ (left). LERS does not control the global error $\|e_h\|_{1,\Omega}$ (right). The optimal error decay is indicated by the dashed line with slope -1.

subdomain Ω_0. Since the subdomain Ω_0 contains the corner, there is a gain of a full power of meshsize of the weighted L^2-estimator with respect to the energy estimator in the complement of the subdomain Ω_1 (see Remark 3.8). A consequence of *equidistribution* of element indicators is that less mesh refinement is needed in the subdomain $\Omega\backslash\Omega_1$, which makes LERS more efficient in reducing $\|e_h\|_{1,\Omega_0}$. In fact, it requires GERS about 28000 DOFs to get the same accuracy in the subdomain Ω_0. This gain becomes more significant asymptotically.

Figure 2 is quite indicative. We know that for a quasi-optimal mesh $\|u - u_h\|_{1,\Omega_0} = CN^{-k/2}$ asymptotically, where N is the number of degrees of freedom, and k is the polynominal degree. Therefore, in the log-log plot the optimal decay of $\|u - u_h\|_{1,\Omega_0}$ is a straight line with slope -1 for quadratic elements. Figure 2 (left) shows that the decay of the local error $\|u - u_h\|_{1,\Omega_0}$ for both methods is about -1 asympotically. Therefore we can conclude that LERS possesses a smaller constant C than GERS but not a faster asymptotic decay.

Since LERS is not designed to control the *global* energy error, its decay deteriorates asymptotically. This effect can be observed in Figure 2 (right).

4.3 Example: Pollution Effect

Consider the same problem as in §4.2, but with a hole $B(x_0, 0.5)$ inside the L-shape domain of center $x_0 = (2,2)$ (see Figure 3). Let $\Omega_0 = B(x_0, 1)$ and $\Omega_1 = B(x_0, 1.5)$. The subdomain Ω_0 does not contain the reentrant corner. We want to examine how pollution errors are controlled adaptively. We still use quadratic elements and Theorem 3.6.

Figure 3 displays the meshes of LERS and GERS with almost the same DOFs of about 19000. Note that these two strategies lead to different local refinements. As explained in Remark 3.9, there is a gain of the fractional power $h_T^{1-\beta}$ near the origin and a full power away from the origin in the subdomain $\Omega\backslash\Omega_1$. As a result of *equidistribution* of element indicators for LERS, we see a

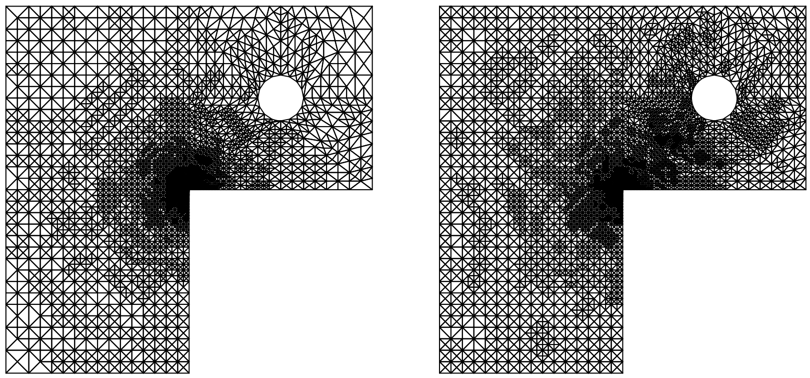

Figure 3: Example 4.3. Meshes of GERS (left) and LERS (right) with DOFs of about 19000 for $\Omega_0 = B(x_0, 1.0) \cap \Omega$ with $x_0 = (2.0, 2.0)$; $\|e_h\|_{\Omega_0} \approx 7.03320103e-5$ for GERS and $\|e_h\|_{\Omega_0} \approx 2.56768096e-5$ for LERS.

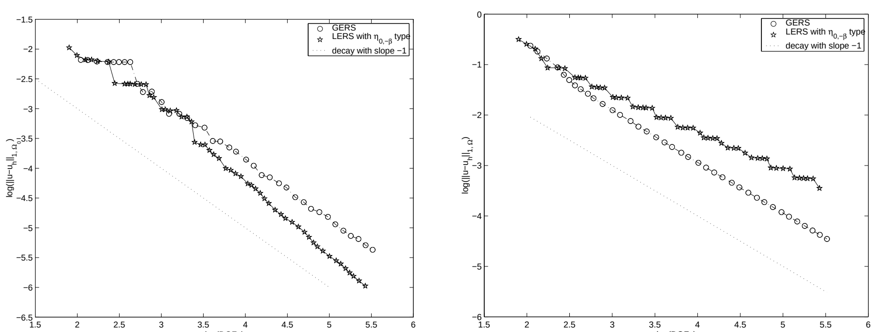

Figure 4: Example 4.3. Local error control of LERS for $\|e_h\|_{1,\Omega_0}$ (left). LERS does not control the global error $\|e_h\|_{1,\Omega}$ (right). The optimal error decay is indicated by the dashed line with slope -1.

better balance between the refinements inside and outside the subdomain Ω_1, with a slightly higher mesh grading inside Ω_1. To get the same accuracy, GERS needs about 55000 DOFS. The improvement due to LERS is thus remarkable.

In Figure 4, we see that the decay of $\|e_h\|_{1,\Omega_0}$ for GERS is still -1, while the decay by LERS is faster in the preasymptotic regime. LERS shows its efficiency in achieving the same local accuracy with much fewer DOFs. Again from Figure 4, we can see that LERS exhibits a bad global error control over $\|u - u_h\|_{1,\Omega}$.

4.4 Example: Interior Layer

The objective of this example is to illustrate the behavior of Theorem 3.14 when studying a problem with a rapidly varying right hand side. For this, we consider a square domain $\Omega = (-2, 2)^2$ with the exact solution given by

$$u = \arctan(5(|x|^2 - 3)),$$

for $A = I$ in (1.1). The boundary data g and right hand side f are defined accordingly. We let $\Omega_0 = B(x_0, 0.7)$ and $\Omega_1 = B(x_0, 1.4)$ with $x_0 = (0, 0)$.

We use quadratic elements and apply GERS and LERS with estimators η_0 and η_{-1} to examine $\|u - u_h\|_{1,\Omega_0}$. Note that due to a regularity result for square domains (see Example 3 in [17]), the conditions for Theorem 3.14 are satisfied. Therefore we can apply (3.29) in Theorem 3.14 even though there are corners.

Figure 5 displays the meshes for these 3 different refinement strategies. From Figure 5 (left) –the mesh generated by GERS– we see that the true solution displays strong interior layer across the circunference $\partial B(x_0, \sqrt{3})$. In contrast, from Figure 5 (right) –the mesh generated by LERS with estimator η_{-1}– we also see an higher mesh grading inside Ω_1, especially across $\partial\Omega_1$. As explained in Remark 3.15, there is a gain of power h_T^2 in the subdomain $\Omega\backslash\Omega_1$, and the *equidistribution* of element indicators of η_1 type in Ω_1 and η_{-1} type in $\Omega\backslash\Omega_1$ results in higher mesh density inside Ω_1 as compared with Figure 5 (left and middle). However, to achieve an accuracy $\|e_h\|_{1,\Omega_0} \approx 5e-4$ the algorithms need

DOFs = 220000 (GERS),

DOFs = 50000 (LERS of η_0 type),

DOFs = 15000 (LERS of η_{-1} type).

This superior performance of LERS in reducing the local error $\|e_h\|_{1,\Omega_0}$ conforms with the well known fact that higher order elements performs more locally. The meshes produced by LERS are again quasi-optimal as documented in Figure 6 (left), with η_{-1} showing a superior performance over η_0.

5 Conclusions

We have derived rigorous and computable *local* residual-type a posteriori error estimators for linear scalar elliptic problems, in the spirit of the interior a priori error estimates of Nitsche and Schatz [17]. The estimators consists of H^1-error indicators in a local region of interest plus global weighted L^2-type error indicators which capture the pollution effect. Numerical experiments show the

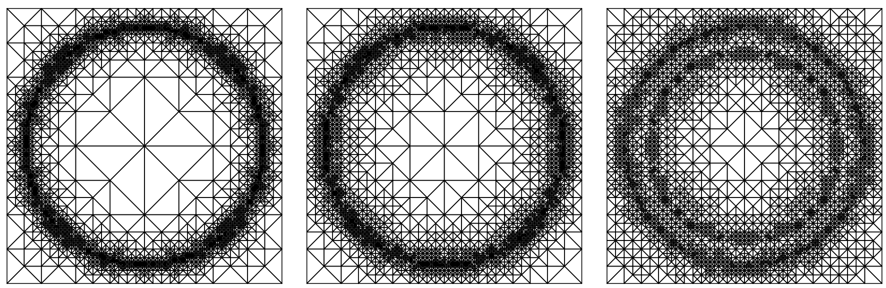

Figure 5: Example 4.4. Meshes of GERS (left), LERS of η_0 type (middle) and LERS of η_{-1} type (right) with DOFs of about 15000 and $\Omega_0 = B(x_0, 0.7)$, $\Omega_1 = B(x_0, 1.4)$, $x_0 = (0,0)$; $\|e_h\|_{1,\Omega_0} \approx 6.70054269e-3$ (left), $\|e_h\|_{1,\Omega_0} \approx 1.42439962e-3$ (middle), and $\|e_h\|_{1,\Omega_0} \approx 4.94662142e-4$ (right).

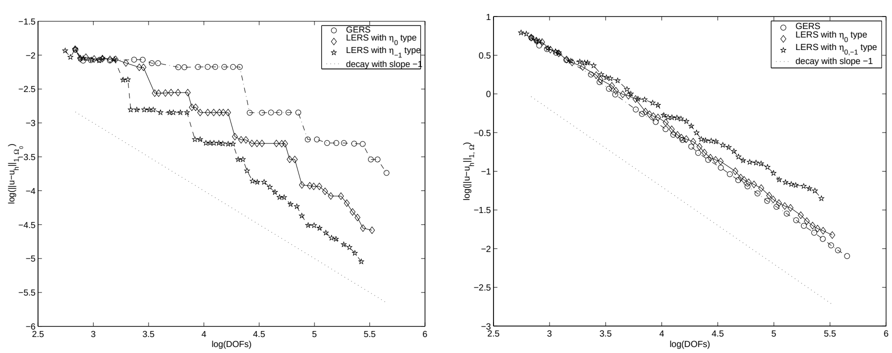

Figure 6: Example 4.4. Quasioptimality of LERS for $\|e_h\|_{1,\Omega_0}$ (left). LERS does not control the global error $\|e_h\|_{1,\Omega}$ (right). The optimal error decay is indicated by the dashed line with slope -1.

superior performance of our adaptive refinement procedure in controling local accuracy with nearly minimal number of degrees of freedom. The ensuing quasi-optimal meshes do not control the global H^1-error though since they are not designed to do so. A possible drawback of our algorithm is the explicit knowledge of geometric singularities of the problem at hand.

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Xiaohai Liao
Department of Mathematics, University of Maryland, College Park, MD 20742, USA.
Email: `xxl@math.umd.edu`.

Ricardo H. Nochetto
Department of Mathematics and Institute for Physical Science and Technology, University of Maryland, College Park, MD 20742, USA.
Partially supported by NSF Grants DMS-9623394 and DMS-9971450.
Email: `rhn@math.umd.edu`.
`http://www.math.umd.edu/~rhn`.