OPTIMAL RELAXATION PARAMETER FOR THE UZAWA METHOD

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ABSTRACT. We consider the Uzawa method to solve the stationary Stokes equations discretized with stable finite elements. An iteration step consists of a velocity update $\mathbf{u}^{n+1}$ involving the (augmented Lagrangian) operator $-\nu\Delta - \rho\nabla\mathrm{div}$ with $\rho \geq 0$, followed by the pressure update $p^{n+1} = p^n - \alpha\nu\mathrm{div}\ \mathbf{u}^{n+1}$, the so-called Richardson update. We prove that the inf-sup constant β satisfies $\beta \leq 1$ and that, if $\sigma = 1 + \rho\nu^{-1}$, the iteration converges linearly with a contraction factor $\beta^2\alpha\sigma^{-1}(2\sigma - \alpha)$ provided $0 < \alpha < 2\sigma$. This yields the optimal value $\alpha = \sigma$ regardless of β.

1. INTRODUCTION

Given an open bounded polygon Ω in $\mathbb{R}^d$, with $d \geq 2$, we consider the stationary Stokes equations, namely the simplest model for incompressible viscous flows:

$$-\nu\triangle\mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega, \tag{1.1}$$

$$\mathrm{div}\ \mathbf{u} = 0, \quad \text{in } \Omega, \tag{1.2}$$

with vanishing Dirichlet boundary condition $\mathbf{u} = \mathbf{0}$ on $\partial\Omega$ and pressure mean-value $\int_\Omega p = 0$. Here the unknowns are the (vector) velocity field $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and the (scalar) pressure $p \in L_0^2(\Omega)$; the forcing function satisfies $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\nu = Re^{-1}$ is the reciprocal of the Reynolds number. In view of the incompressibility constraint (1.2), the momentum equation (1.1) is equivalent to the *augmented Lagrangian* formulation with $\rho \geq 0$

$$-\nu\triangle\mathbf{u} - \rho\nabla\mathrm{div}\ \mathbf{u} + \nabla p = \mathbf{f}, \quad \text{in } \Omega.$$

This equivalence is no longer true at the discrete level, where the additional operator $-\rho\nabla\mathrm{div}$ may improve the convergence of iterative methods [3]. We provide a quantitative measure of such improvement in this paper.

The following (infinite-dimensional) Uzawa algorithm to solve the Stokes system is known to converge for appropriate values of the *relaxation parameter* α [2, 3, 4, 5, 6].

Algorithm 1 (Uzawa Method). *Given a suitable relaxation parameter $\alpha > 0$ and initial guess p^0:*

Step 1: *Find $\mathbf{u}^{n+1} \in \mathbf{H}_0^1(\Omega)$ as the solution of*

$$-\nu\triangle\mathbf{u}^{n+1} - \rho\nabla\mathrm{div}\ \mathbf{u}^{n+1} + \nabla p^n = \mathbf{f}, \qquad \text{in } \Omega;$$

Step 2: *Find $p^{n+1} \in L_0^2(\Omega)$ from the Richardson update*

$$p^{n+1} = p^n - \alpha\nu\mathrm{div}\ \mathbf{u}^{n+1}.$$

Convergence of Algorithm 1 for $\rho = 0$ is proved via boundedness and coercivity of the Schur complement operator $S = -\mathrm{div}\,(-\triangle^{-1})\nabla$ with sufficiently small $\alpha < 1$ in [2, 4]. In [6] Temam shows the convergence range $0 < \alpha < 2$ also for $\rho = 0$, but does not quite prove that Algorithm 1 is a contraction and thus cannot find the optimal value of α. For the case $\rho > 0$, Fortin and Glowinski prove convergence for $0 < \alpha \leq 2\rho/\nu$ using a spectral analysis [3].

The choice of relaxation parameter $\alpha > 0$ is crucial for the convergence of Uzawa method because a small value of α yields a large contraction factor whereas a large value may lead to divergence. It is the purpose of this note to show convergence for all $0 < \alpha < 2(1 + \rho/\nu)$ and that $\alpha = 1 + \rho/\nu$ is

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an optimal choice. This has been already instrumental in [1] for $\rho = 0$. Our analysis is in the spirit of that in [6] for $\rho = 0$, but it gives rise to more precise bounds.

We consider now a finite element discretization. Let $\mathfrak{T} = \{K\}$ be a shape-regular partition of Ω of local meshsize h into closed elements K; $\mathfrak{T}$ can be highly graded though. The finite element spaces to be used for approximating the velocity space $\mathbf{H}_0^1(\Omega)$ and pressure space $L_0^2(\Omega)$ are:

$$\mathbb{V}_h := \{\mathbf{v}_h \in \mathbf{H}_0^1(\Omega) : \mathbf{v}_h|_K \in \mathcal{P}(K), \text{ for all } K \in \mathfrak{T}\},$$
$$\mathbb{P}_h := \{\mathbf{v}_h \in L^2(\Omega) : p_h|_K \in \mathcal{Q}(K), \text{ for all } K \in \mathfrak{T}\},$$

where $\mathcal{P}(K)$ and $\mathcal{Q}(K)$ are spaces of polynomials with degree bounded uniformly with respect to $K \in \mathfrak{T}$ [2, 4]. These spaces are compatible, namely they satisfy the following discrete *inf-sup condition: There exists a constant $\beta > 0$ such that* [2, 4]

$$\inf_{p_h \in \mathbb{P}_h} \sup_{\mathbf{v}_h \in \mathbb{V}_h} \frac{\langle \operatorname{div} \mathbf{v}_h \, , \, p_h \rangle}{\|\nabla \mathbf{v}_h\| \|p_h\|} \geq \beta; \tag{1.3}$$

hereafter $\|\cdot\|$ indicates the L^2-norm in Ω. Hence, there is a unique solution $(\mathbf{u}_h, p_h) \in \mathbb{V}_h \times \mathbb{P}_h$ to the following discrete Stokes problem [2, 4]:

$$\begin{aligned} \nu \langle \nabla \mathbf{u}_h \, , \, \nabla \mathbf{w}_h \rangle - \langle p_h \, , \, \operatorname{div} \mathbf{w}_h \rangle = \langle \mathbf{f} \, , \, \mathbf{w}_h \rangle \, , \qquad & \forall \mathbf{w}_h \in \mathbb{V}_h, \\ \langle \operatorname{div} \mathbf{u}_h \, , \, q_h \rangle = 0, \qquad & \forall q_h \in \mathbb{P}_h. \end{aligned} \tag{1.4}$$

Proposition 1 (Inf-Sup Constant). *Let β be the inf-sup constant of* (1.3). *Then we have*

$$\beta \leq 1. \tag{1.5}$$

The discrete Uzawa method, a discrete version of Algorithm 1, is known to be an effective iteration to compute $(\mathbf{u}_h, p_h)$, and reads as follows [2, 4].

Algorithm 2 (Discrete Uzawa Method). *For a suitable $\alpha > 0$ and initial guess $p_h^0 \in \mathbb{P}_h$:*

Step 1: *Find $\mathbf{u}_h^{n+1} \in \mathbb{V}_h$ as the solution of*

$$\nu \left\langle \nabla \mathbf{u}_h^{n+1} \, , \, \nabla \mathbf{w}_h \right\rangle + \rho \left\langle \operatorname{div} \mathbf{u}_h^{n+1} \, , \, \operatorname{div} \mathbf{w}_h \right\rangle - \langle p_h^n \, , \, \operatorname{div} \mathbf{w}_h \rangle = \langle \mathbf{f} \, , \, \mathbf{w}_h \rangle \, , \qquad \forall \mathbf{w}_h \in \mathbb{V}_h; \tag{1.6}$$

Step 2: *Find $p_h^{n+1} \in \mathbb{P}_h$ from the Richardson update*

$$\left\langle p_h^{n+1} \, , \, q_h \right\rangle = \langle p_h^n \, , \, q_h \rangle - \alpha\nu \left\langle \operatorname{div} \mathbf{u}_h^{n+1} \, , \, q_h \right\rangle . \tag{1.7}$$

In §§3 and 4, we prove the following sharp decay estimates.

Theorem 1 (Convergence Rate for Pressure). *If $0 < \alpha < 2\sigma$, then Algorithm 2 satisfies*

$$\|p_h - p_h^{n+1}\| \leq \left(1 - \alpha\beta^2\sigma^{-2} \left(2\sigma - \alpha\right)\right)^{1/2} \|p_h - p_h^n\|, \tag{1.8}$$

where $\sigma := 1 + \frac{\rho}{\nu}$. The same estimate is valid for Algorithm 1.

Corollary 1 (Convergence Rate for Velocity). Both Algorithms 1 and 2 satisfy

$$\left\|\nabla(\mathbf{u}_h - \mathbf{u}_h^{n+1})\right\| \leq \nu^{-1} \left(1 - \alpha\beta^2\sigma^{-2} \left(2\sigma - \alpha\right)\right)^{n/2} \left\|p_h - p_h^0\right\|. \tag{1.9}$$

Remark 1.1 (Optimal relaxation Parameter). Consider now the function $f(\alpha) = \left(1 - \frac{\alpha\beta^2}{\sigma^2}(2\sigma - \alpha)\right)$. We see that Algorithm 2 converges *linearly* with contraction factor $0 < f(\alpha) < 1$ provided $0 < \alpha < 2\sigma$. Since the minimum of $f(\alpha)$ is $1 - \beta^2$ at $\alpha = \sigma$, we conclude that the *optimal* value of α is

$$\alpha = 1 + \frac{\rho}{\nu}.$$

We observe that this result is independent of the domain Ω, and valid for both Algorithms 1 and 2, whereas the eigenvalues of the Schur complement operator, the discrete version of $S := \operatorname{div} (\triangle + \rho\nu^{-1}\nabla \operatorname{div})^{-1}\nabla$, depend on Ω. It is plausible that for a given Ω and finite element pair $(\mathbb{V}_h, \mathbb{P}_h)$, a special analysis would yield a better value for α since Uzawa is simply a Richardson iteration for the Schur complement. It is also plausible that for a rectangular domain with high aspect ratio, $\alpha = \sigma$ is the only choice valid for all aspect ratios. This deserves further investigation.

We also point out that (1.9) improves upon [6], where $\mathbf{u}^n$ is shown to converge weakly in $\mathbf{H}_0^1(\Omega)$.

2. Proof of Proposition 1

In this section, we prove a couple of crucial properties of the divergence operator, in particular an upper bound for the inf-sup constant β of (1.3). Since the following known result plays a pivotal role in our subsequent discussion, we present its elementary proof; we refer to [6, p.140].

Lemma 2.1 (Div-Grad Relation)**.** *For all* $\mathbf{v} \in \mathbf{H}_0^1(\Omega)$, *we have*

$$\|\text{div } \mathbf{v}\| \le \|\nabla \mathbf{v}\|. \tag{2.1}$$

Proof. Given $\mathbf{v} = (v_i)_{i=1}^d \in \mathbf{H}_0^1(\Omega)$, there exists a sequence $\{\mathbf{v}^n\} \in \mathbf{C}_0^\infty(\Omega)$ such that

$$\|\nabla(\mathbf{v}^n - \mathbf{v})\| \to 0 \qquad \text{as } n \to \infty. \tag{2.2}$$

Since $\mathbf{v}^n \in \mathbf{C}_0^\infty(\Omega)$, integration by parts implies

$$\begin{aligned}
\|\text{div } \mathbf{v}^n\|^2 &= \int_\Omega \left(\sum_{i=1}^d \partial_{x_i} v_i^n \right)^2 d\mathbf{x} \\
&= \int_\Omega \sum_{i=1}^d \sum_{j=1}^d \partial_{x_i} v_i^n \partial_{x_j} v_j^n d\mathbf{x} \\
&= \int_\Omega \sum_{i=1}^d \sum_{j=1}^d \partial_{x_i} v_j^n \partial_{x_j} v_i^n d\mathbf{x} \\
&\le \int_\Omega \sum_{i=1}^d \sum_{j=1}^d (\partial_{x_i} v_j^n)^2 d\mathbf{x} = \|\nabla \mathbf{v}^n\|^2.
\end{aligned}$$

The assertion (2.1) follows from (2.2) upon passing to the limit $n \to \infty$. ∎

Applying Lemma 2.1, we can find an upper bound of the inf-sup constant β of (1.3).

Proof of Proposition 1. Let $q_h \in \mathbb{P}_h$ be an arbitrary function. Then, the discrete inf-sup condition (1.3) is equivalent to the existence of a function $\mathbf{v}_h \in \mathbb{V}_h$ such that [2, 4]

$$\langle \text{div } \mathbf{v}_h \,,\, q_h \rangle = \|q_h\|^2 \qquad \text{and} \qquad \|\nabla \mathbf{v}_h\| \le \frac{1}{\beta}\|q_h\|. \tag{2.3}$$

Then, by virtue of (2.3) and Lemma 2.1, we obtain

$$\|q_h\|^2 = \langle \text{div } \mathbf{v}_h \,,\, q_h \rangle \le \|\text{div } \mathbf{v}_h\| \; \|q_h\| \le \|\nabla \mathbf{v}_h\| \; \|q_h\| \le \frac{1}{\beta}\|q_h\|^2,$$

which implies the asserted estimate (1.5). ∎

3. Proof of Theorem 1: Case $\rho = 0$

In this section we prove Theorem 1 for $\rho = 0$. To this end, we use the following error functions:

$$\mathbf{E}_h^{n+1} = \mathbf{u}_h - \mathbf{u}_h^{n+1}, \qquad e_h^{n+1} = p_h - p_h^{n+1}.$$

We proceed in several steps. Upon subtracting (1.6) (with $\rho = 0$) from (1.4), we have

$$\nu \left\langle \nabla \mathbf{E}_h^{n+1} \,,\, \nabla \mathbf{w}_h \right\rangle - \langle e_h^n \,,\, \text{div } \mathbf{w}_h \rangle = 0, \qquad \forall \mathbf{w}_h \in \mathbb{V}_h. \tag{3.1}$$

Since $\mathbf{u}_h$ is a discrete divergence free function, (1.7) can be written by

$$\left\langle e_h^{n+1} \,,\, q_h \right\rangle = \langle e_h^n \,,\, q_h \rangle - \alpha\nu \left\langle \text{div } \mathbf{E}_h^{n+1} \,,\, q_h \right\rangle \qquad \forall q_h \in \mathbb{P}_h. \tag{3.2}$$

If we choose $\mathbf{w}_h = \mathbf{E}_h^{n+1}$, then (3.1) becomes

$$\nu \left\|\nabla \mathbf{E}_h^{n+1}\right\|^2 - \left\langle e_h^n \,,\, \text{div } \mathbf{E}_h^{n+1} \right\rangle = 0. \tag{3.3}$$

In light of (3.2) and (3.3), we obtain

$$\begin{aligned}\nu\|\nabla\mathbf{E}_h^{n+1}\|^2 &= -\frac{1}{\alpha\nu}\left\langle e_h^n\,,\,e_h^{n+1}-e_h^n\right\rangle \\ &= -\frac{1}{2\alpha\nu}\left(\|e_h^{n+1}\|^2-\|e_h^n\|^2-\|e_h^{n+1}-e_h^n\|^2\right).\end{aligned}\tag{3.4}$$

Consequently, we arrive at

$$2\alpha\nu^2\|\nabla\mathbf{E}_h^{n+1}\|^2+\|e_h^{n+1}\|^2=\|e_h^n\|^2+\|e_h^{n+1}-e_h^n\|^2.\tag{3.5}$$

The next step is to estimate $\|e_h^{n+1}-e_h^n\|^2$. Choosing $q_h=e_h^{n+1}-e_h^n$ in (3.2) gives

$$\|e_h^{n+1}-e_h^n\|^2=-\alpha\nu\left\langle \text{div }\mathbf{E}_h^{n+1}\,,\,e_h^{n+1}-e_h^n\right\rangle\le\alpha\nu\|\text{div }\mathbf{E}_h^{n+1}\|\;\|e_h^{n+1}-e_h^n\|,$$

whence

$$\|e_h^{n+1}-e_h^n\|\le\alpha\nu\|\text{div }\mathbf{E}_h^{n+1}\|\le\alpha\nu\|\nabla\mathbf{E}_h^{n+1}\|.\tag{3.6}$$

Replacing (3.6) into (3.5) immediately leads to

$$\nu^2\alpha(2-\alpha)\|\nabla\mathbf{E}_h^{n+1}\|^2+\|e_h^{n+1}\|^2\le\|e_h^n\|^2.\tag{3.7}$$

We finally prove (1.8) for $\rho=0$. Since $e_h^n\in\mathbb{P}_h$, there exists a function $\mathbf{v}_h\in\mathbb{V}_h$ such that

$$\langle\text{div }\mathbf{v}_h\,,\,e_h^n\rangle=\|e_h^n\|^2\qquad\text{and}\qquad\|\nabla\mathbf{v}_h\|\le\frac{1}{\beta}\|e_h^n\|,\tag{3.8}$$

which is equivalent to (1.3) [2, 4]. In view of (3.1) and (3.8), we get

$$\begin{aligned}\|e_h^n\|^2&=\langle\text{div }\mathbf{v}_h\,,\,e_h^n\rangle\\&=\nu\left\langle\nabla\mathbf{E}_h^{n+1}\,,\,\nabla\mathbf{v}_h\right\rangle\\&\le\nu\|\nabla\mathbf{E}_h^{n+1}\|\|\nabla\mathbf{v}_h\|\\&\le\frac{\nu}{\beta}\|\nabla\mathbf{E}_h^{n+1}\|\|e_h^n\|.\end{aligned}$$

Consequently,

$$\beta\|e_h^n\|\le\nu\|\nabla\mathbf{E}_h^{n+1}\|.\tag{3.9}$$

In light of (3.7) and (3.9), we easily obtain

$$\beta^2\alpha(2-\alpha)\|e_h^n\|^2+\|e_h^{n+1}\|^2\le\|e_h^n\|^2,$$

which implies (1.8) for $\rho=0$, and thus completes the proof of Theorem 1.

On the other hand, Corollary 1 follows from (3.3) together with Proposition 1 and Theorem 1.

4. Proof of Theorem 1: Case $\rho > 0$

In this section, we prove Theorem 1 for $\rho>0$. We note that both (3.2) and (3.6) are still valid for $\rho>0$, because they are solely based on the Richardson update (1.7). Upon subtracting (1.6) (with $\rho>0$) from (1.4), we have

$$\nu\left\langle\nabla\mathbf{E}_h^{n+1}\,,\,\nabla\mathbf{w}_h\right\rangle+\rho\left\langle\text{div }\mathbf{E}_h^{n+1}\,,\,\text{div }\mathbf{w}_h\right\rangle-\langle e_h^n\,,\,\text{div }\mathbf{w}_h\rangle=0,\qquad\forall\mathbf{w}_h\in\mathbb{V}_h.\tag{4.1}$$

Choosing $\mathbf{w}_h=\mathbf{E}_h^{n+1}$ and applying (3.2), (4.1) becomes

$$\begin{aligned}\nu\|\nabla\mathbf{E}_h^{n+1}\|^2+\rho\|\text{div }\mathbf{E}_h^{n+1}\|^2&=\langle e_h^n\,,\,\text{div }\mathbf{E}_h\rangle\\&=-\frac{1}{\alpha\nu}\left\langle e_h^n\,,\,e_h^{n+1}-e_h^n\right\rangle\\&=-\frac{1}{2\alpha\nu}\left(\|e_h^{n+1}\|^2-\|e_h^n\|^2-\|e_h^{n+1}-e_h^n\|^2\right),\end{aligned}$$

instead of (3.4). In light of (3.6), we thus obtain

$$2\alpha\nu^2\left(\left\|\nabla \mathbf{E}_h^{n+1}\right\|^2+\frac{\rho}{\nu}\left\|\operatorname{div}\,\mathbf{E}_h^{n+1}\right\|^2-\frac{\alpha}{2}\left\|\operatorname{div}\,\mathbf{E}_h^{n+1}\right\|^2\right)+\left\|e_h^{n+1}\right\|^2\le\|e_h^n\|^2. \tag{4.2}$$

Replacing now

$$-\frac{\alpha}{2}\left\|\operatorname{div}\,\mathbf{E}_h^{n+1}\right\|^2=-\frac{\alpha\nu}{2\rho}\left(\left\|\nabla \mathbf{E}_h^{n+1}\right\|^2+\frac{\rho}{\nu}\left\|\operatorname{div}\,\mathbf{E}_h^{n+1}\right\|^2\right)+\frac{\alpha\nu}{2\rho}\left\|\nabla \mathbf{E}_h^{n+1}\right\|^2,$$

into (4.2), we end up with

$$2\alpha\nu^2\left(1-\frac{\alpha\nu}{2\rho}\right)\left(\left\|\nabla \mathbf{E}_h^{n+1}\right\|^2+\frac{\rho}{\nu}\left\|\operatorname{div}\,\mathbf{E}_h^{n+1}\right\|^2\right)+\frac{\alpha^2\nu^3}{\rho}\left\|\nabla \mathbf{E}_h^{n+1}\right\|^2+\left\|e_h^{n+1}\right\|^2\le\|e_h^n\|^2. \tag{4.3}$$

Using inf-sup property (3.8), we deduce

$$\begin{aligned}
\|e_h^n\|^2 &= \langle \operatorname{div}\,\mathbf{v}_h\,,\,e_h^n\rangle\\
&= \nu\left\langle \nabla \mathbf{E}_h^{n+1}\,,\,\nabla\mathbf{v}_h\right\rangle+\rho\left\langle \operatorname{div}\,\mathbf{E}_h^{n+1}\,,\,\operatorname{div}\,\mathbf{v}_h\right\rangle\\
&\le \nu\left\|\nabla \mathbf{E}_h^{n+1}\right\|\|\nabla\mathbf{v}_h\|+\rho\left\|\operatorname{div}\,\mathbf{E}_h^{n+1}\right\|\|\operatorname{div}\,\mathbf{v}_h\|\\
&\le \left(\nu\|\nabla\mathbf{v}_h\|^2+\rho\|\operatorname{div}\,\mathbf{v}_h\|^2\right)^{\frac12}\left(\nu\left\|\nabla \mathbf{E}_h^{n+1}\right\|^2+\rho\left\|\operatorname{div}\,\mathbf{E}_h^{n+1}\right\|^2\right)^{\frac12}\\
&\le \frac{\nu\sqrt{\sigma}}{\beta}\|\nabla e_h^n\|\left(\left\|\nabla \mathbf{E}_h^{n+1}\right\|^2+\frac{\rho}{\nu}\left\|\operatorname{div}\,\mathbf{E}_h^{n+1}\right\|^2\right)^{\frac12},
\end{aligned}$$

where we have used (2.1) for $\mathbf{v}=\mathbf{v}_h$ and $\sigma=1+\frac{\rho}{\nu}$. Consequently, we infer that

$$\|e_h^n\|^2\le\frac{\nu^2\sigma}{\beta^2}\left(\left\|\nabla \mathbf{E}_h^{n+1}\right\|^2+\frac{\rho}{\nu}\left\|\operatorname{div}\,\mathbf{E}_h^{n+1}\right\|^2\right)$$

and, making use of (2.1) again for $\mathbf{v}=\mathbf{E}_h^{n+1}$, that

$$\|e_h^n\|^2\le\frac{\nu^2\sigma^2}{\beta^2}\left\|\nabla \mathbf{E}_h^{n+1}\right\|^2.$$

Replacing these two inequalities into (4.3), we obtain

$$\left(\frac{2\alpha\beta^2}{\sigma}\left(1-\frac{\alpha\nu}{2\rho}\right)+\frac{\alpha^2\beta^2\nu}{\rho\sigma^2}\right)\|e_h^n\|^2+\left\|e_h^{n+1}\right\|^2\le\|e_h^n\|^2,$$

or equivalently,

$$\left\|e_h^{n+1}\right\|^2\le\left(1-\frac{\alpha\beta^2}{\sigma^2}\left(2\sigma-\frac{\alpha\nu}{\rho}(\sigma-1)\right)\right)\|e_h^n\|^2=\left(1-\frac{\alpha\beta^2}{\sigma^2}(2\sigma-\alpha)\right)\|e_h^n\|^2.$$

This is the asserted estimate (1.8). The proof of Theorem 1 for $\rho>0$ is thus complete. Finally, the proof of Corollary 1 for $\rho>0$ is identical to the case $\rho=0$.

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