Equivalence Principle for Optimization of Sparse Versus Low-Spread Representations for Signal Estimation in Noise

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ABSTRACT: Estimation of a sparse signal representation, one with the minimum number of nonzero components, is hard. In this paper, we show that for a nontrivial set of the input data the corresponding optimization problem is equivalent to and can be solved by an algorithm devised for a simpler optimization problem. The simpler optimization problem corresponds to estimation of signals under a low-spread constraint. The goal of the two optimization problems is to minimize the Euclidian norm of the linear approximation error with an l^p penalty on the coefficients, for $p = 0$ (sparse) and $p = 1$ (low-spread), respectively. The l^0 problem is hard, whereas the l^1 problem can be solved efficiently by an iterative algorithm. Here we precisely define the l^0 optimization problem, construct an associated l^1 optimization problem, and show that for a set with open interior of the input data the optimizers of the two optimization problems have the same support. The associated l^1 optimization problem is used to find the support of the l^0 optimizer. Once the support of the l^0 problem is known, the actual solution is easily found by solving a linear system of equations. However, we point out our approach does not solve the harder optimization problem for all input data and thus may fail to produce the optimal solution in some cases. Int J Imaging Syst Technol, 15, 10–17, 2005; Published online in Wiley InterScience (www.interscience.wiley.com). DOI 10.1002/ima.20034

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I. INTRODUCTION

Consider the following linear problem: Given $\mathbf{x} \in \mathbf{C}^n$ of the form

$$\mathbf{x} = \mathbf{A}\mathbf{s} + v, \tag{1}$$

estimate $\mathbf{s} \in \mathbf{C}^n$, where $\mathbf{A}$ is a given $n \times n$ invertible (complex) matrix and $v \in \mathbf{C}^n$ is an interference (noise) term. Obviously, when $v = 0$ the solution of this problem is trivially, $\mathbf{s} = \mathbf{A}^{-1}\mathbf{x}$. However, in a practical setting $v \neq 0$ and it may also happen that $\mathbf{A}$ is ill-conditioned in which case the inversion becomes a problem. Two approaches have been devised to deal with these issues.

One approach is purely deterministic and addresses mainly the case when $\mathbf{A}$ is ill-conditioned. The main observation in this case is that inverting $\mathbf{A}$ yields

$$\mathbf{A}^{-1}\mathbf{x} = \mathbf{s} + \mathbf{A}^{-1}v,$$

which potentially amplifies the noise-like error in the data. The solution is then to minimize a criterion containing two terms: one term that measures how well $\mathbf{x}$ matches $\mathbf{A}\mathbf{s}$ without regard to the noise-term and a second term that penalizes large entries in $\mathbf{s}$, which, potentially, are due to amplified noise. Thus, the regularized problem becomes

$$\min_{\mathbf{s}\in\mathbf{C}^n} |||\mathbf{A}\mathbf{s} - \mathbf{x}|||_1 + \lambda|||\mathbf{s}|||_2, \tag{2}$$

where $|||\cdot|||_1$ and $|||\cdot|||_2$ are some norm-like measures chosen more often from a convenient algebraic computation point of view. One of the most popular choices for these measures is the square of the Euclidian norm $\|\cdot\|$, $\|\mathbf{y}\| = (\sum_{k=1}^{n} |y_k|^2)^{1/2}$, thus the problem can be stated as

$$\min_{\mathbf{s}\in\mathbf{C}^n} \|\mathbf{A}\mathbf{s} - \mathbf{x}\|^2 + \lambda\|\mathbf{s}\|^2, \tag{3}$$

where λ is a regularization parameter. Choosing $\|\cdot\|^2$ for $|||\cdot|||_1$ and $|||\cdot|||_2$ as in (3) results in the Tikhonov regularization method (Engl et al., 1996). Alternatively, and more general, one can use the following measures:

$$\min_{\mathbf{s}\in\mathbf{C}^n} \|\mathbf{A}\mathbf{s} - \mathbf{x}\|^2 + \lambda\|\mathbf{s}\|_p^p \quad \text{for } p > 0 \tag{4}$$

and

$$\min_{\mathbf{s}\in\mathbf{C}^n} \|\mathbf{A}\mathbf{s} - \mathbf{x}\|^2 + \lambda\|\mathbf{s}\|_0 \quad \text{for } p = 0, \tag{5}$$

where $\|\cdot\|_p$ is the p (quasi) norm defined by

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$$\|\mathbf{y}\|_p = \left(\sum_{k=1}^{n} |y_k|^p\right)^{1/p} \quad \text{for } p > 0$$

and

$$\|\mathbf{y}\|_0 = |\text{supp}(\mathbf{y})| \quad \text{for } p = 0,$$

with $\|\cdot\|_0$, instead of $\|\cdot\|_0^0$, because $\lim_{p\searrow 0}\|s\|_p^p = \|s\|_0$, $\text{supp}(\mathbf{y}) = \{k|y_k \neq 0\}$ is the support of $\mathbf{y}$, and $|S|$ denotes the cardinal of the discrete set S. For $0 < p < 1$, $\|\cdot\|_p$ defines a quasi-norm, whereas for $p = 0$ it is not even linear with respect to scalar multiplication.

A second approach to solving (1) uses a stochastic estimation framework. For example, we may assume $\mathbf{v}$ is $N(0, \sigma^2\mathbf{I})$ Gaussian noise and that the signal s has n independent components with a priori distribution Exp($0,p,\mu$) given by

$$f_{\mathbf{s}}(\mathbf{s}) = \prod_{k=1}^{n} C_{p,\mu}\, e^{-|s_k|^p/\mu}. \tag{6}$$

Such distributions, for instance, have been cited as sparse distributions in the work of Zibulevski and Pearlmutter (2001) and Karvanen and Cichocki (2003), since they have a sharper peak than the Gaussian distribution for $p < 2$. It is not hard to show that the maximum a posteriori (MAP) estimator of s in this case is given by

$$\hat{\mathbf{s}}\,\underset{\mathbf{s}\in\mathbf{C}^n}{\text{argmin}}\, \|\mathbf{As} - \mathbf{x}\|^2 + \frac{1}{\sigma^2\mu}\|\mathbf{s}\|_p^p, \tag{7}$$

which is exactly the same as the regularization mentioned before.

The purpose of this paper is to connect the optimization problem with the general form expressed in (7) for $p = 0$ to that with $p = 1$. For $p = 0$, the problem can be simply stated as

$$\min_{\mathbf{s}\in\mathbf{C}^n} \|\mathbf{As} - \mathbf{x}\|^2 + \mu|\{k;\ s_k \neq 0\}|, \tag{8}$$

for some fixed $\mu > 0$. If one is given the support of the optimizer, then finding the optimizer becomes a simple least square problem, and this involves merely solving a linear system. Hence the hard problem is to find the right support. For $p = 1$, the situation is completely different. In the literature, algorithms have been proposed by Engl et al. (1996) and Daubechies et al. (2003) to solve

$$\min_{\mathbf{s}\in\mathbf{C}^n} \|\mathbf{Bx} - \mathbf{s}\|^2 + \lambda\sum_{k=1}^{n}|s_k|, \tag{9}$$

and they converge quickly to a solution. In this paper, we show how the solution of (9) can be used to obtain the solution of (8), for specific choices of $\mathbf{B}$, and λ, and for an open set of data $\mathbf{x}$.

The next section briefly discusses work related to problems (8) and (9). Then Section III presents the main theoretical results, which were grouped together in order to offer a succinct view of the work. Section IV contains proofs of the main lemmas and the central theorem. Section V demonstrates the application of this work in a simple example. Section VI summarizes this work.

II. RELATED WORK

The seminal work by Donoho and Huo (2001), in which a connection of a similar nature has been made, subsequently sparked interest from other researchers (Elad and Bruckstein, 2002; Fuchs, 2002; Gribonval and Nielsen, 2002; Feuer and Nemirovski, 2003; Malioutov et al., 2004). These authors tackled the problem, given a redundant dictionary $D = \{\mathbf{d}_1, \ldots, \mathbf{d}_m\}$ in $\mathbf{C}^n$ ($m > n$), and a vector $\mathbf{x} \in \mathbf{C}^n$ that admits a sparse representation, find the sparse decomposition of $\mathbf{x}$, that is $\mathbf{x} = \sum_{j\in\mathbf{J}} s_j\mathbf{d}_j$ where $|\mathbf{J}| < n$.

The problem can be turned into an optimization problem as follows:

$$\underset{\mathbf{s}\in\mathbf{C}^m \text{ and } \mathbf{D_s}=\mathbf{x}}{\text{argmin}} \{|\text{supp}(\mathbf{s})|\}, \tag{10}$$

where $\mathbf{D} = [\mathbf{d}_1|\cdots|\mathbf{d}_m]$ is the $n \times m$ matrix whose columns are the $\mathbf{d}_j$ vectors.

The main result in the work of Donoho and Huo (2001) is that for a "thin" set of input data $\mathbf{x}$ ("thin" in the sense that is has empty interior), the solution of this problem coincides with the solution of a similar, but easier to solve optimization problem

$$\underset{\mathbf{s}\in\mathbf{C}^m \text{ and } \mathbf{D_s}=\mathbf{x}}{\text{argmin}} \{\|\mathbf{s}\|_1\}. \tag{11}$$

In this framework, our problem can be restated as follows. Define the $n \times 2n$ matrix

$$\mathbf{E} = [A\ I]. \tag{12}$$

Then (7) is equivalent to the following optimization problem

$$\underset{\mathbf{u}\in\mathbf{C}^{2n} \text{ and } \mathbf{Eu}=\mathbf{x}}{\text{argmin}} \|\mathbf{u}\|_{p,2}, \tag{13}$$

where

$$\|\mathbf{u}\|_{p,2} = \sum_{k=1}^{n}|u_k|^p + \sum_{k=n+1}^{2n}|u_k|^2, \tag{14}$$

which is slightly different than (11), even with $p = 1$. Note, $\|\mathbf{u}\|_{p,2}$ is not a norm (or quasi-norm), since it does not scale properly. Unlike the work of Donoho and Huo (2001), our theorem says that for a nonthin set (i.e., with nonempty interior) of input data x, the support of the first n components of the optimizer for $p = 1$ coincides with the support of the first n components of one optimizer for $p = 0$. On the one hand, the conclusion of our theorem is weaker than that of Donoho and Huo (2001), namely the optimizers for $p = 0$ and $p = 1$ do not coincide, but only their supports coincide. On the other hand, our result has a much wider applicability, since the set of input data where the conclusion holds true has nonempty interior.

Optimization problems of type (8) have been analyzed from a computational complexity point of view. More specifically in the work of Davis et al. (1997), the authors proved that when $\mathbf{A}$ is a full rank $n \times m$ matrix with $m = O(n^k)$, for some $k \geq 1$, the finite-input L-term ε-approximation of the optimum value problem (i.e., within ε of the optimum) is NP-complete. The proof though relies crucially upon the redundancy of the dictionary formed by the columns of matrix $\mathbf{A}$. For instance, when $\mathbf{A} = UD$ with U a unitary (or orthogonal) matrix and D a diagonal invertible matrix, we have

$$\|\mathbf{A}\mathbf{s} - \mathbf{x}\|^2 + \|\mathbf{s}\|_0 = \|D\mathbf{s} - U^*\mathbf{x}\|^2 + \|D\mathbf{s}\|_0, \tag{15}$$

which can easily be solved in order $O(n^2)$ time. However, for general **A**, we do not know whether the L-term approximation problem within ε to the optimal value is NP-complete or not.

Research into practical problems leading to similar optimization problems appears in the literature on speech enhancement and image processing, and more generally signal transformation using the independent component analysis (ICA) and blind source separation (BSS) techniques. In speech enhancement, the interest is to use a signal-adapted (i.e., learned from the data) representation, instead of the standard frequency-domain representation, in hope of transforming the signal into a sparse form, which can offer simplification of the complex estimation problems to be dealt with. Recently, many other signal transformation problems seem to benefit from the use of data-dependent transformations, for example independent wavelet bases or independent components learned from the data, in contrast to the use of fixed transformations such as a frequency domain data-independent transformation. One outstanding research question is whether real data in various domains (MRI, EEG, vision, and speech) is amenable to such approaches. Experimental evidence is constantly being gained in this sense. The definition of a sparse representation of a signal here is that a ''small'' number of coefficients different from zero are necessary in a decomposition of the signal using the bases (Zibulevski and Pearlmutter, 2001). The idea of sparse coding is summarized by Hyvarinen et al. (2001).

For example, speech is a sparse signal, and the property has been exploited in the ICA–BSS community for parameter estimation and source separation (Huang et al., 1995; Aoki et al., 2001; Zibulevski and Pearlmutter, 2001). A time-frequency (TF) sparseness assumption has been introduced (Jourjine et al., 2000) and subsequently used in the works of Rickard et al. (2001) and Balan et al. (2003), which allows for the separation of more than two sources given just two mixtures. This sparseness property, called *W-disjoint orthogonality* (WDO), assumes that the signals have nonoverlapping TF representation supports. Given source TF representations $S_1(\omega, t), \ldots, S_N(\omega, t)$, the WDO assumption can be stated as

$$S_i(\omega, t)\, S_j(\omega, t) = 0 \quad \forall i \neq j \quad \forall(\omega, t). \tag{16}$$

This assumption has been shown to be approximately true for speech signals (Rickard and Yilmaz, 2002). Further, WDO is approximately satisfied when one assumes a signal model of the form

$$S(\omega, t) = B(\omega, t)\, G(\omega, t), \tag{17}$$

where $B(\omega, t)$ is a Bernoulli random variable (i.e., it takes a value of only 0 or 1), and $G(\omega, t)$ is a continuously distributed random variable (Balan et al., 2003). It follows that the joint distribution is

$$\begin{aligned} ps_1, s_2(S_1, S_2) &= (1-q)^2\delta(S_1)\delta(S_2) + \\ &q(1-q)(\delta(S_1)p(S_2) + \delta(S_2)p(S_1)) + q^2 p(S_1)p(S_2). \end{aligned} \tag{18}$$

Sparse decompositions directly lead to solving a problem equivalent to (7) in the context of learning a signal dictionary, that is the matrix **A**, such as wavelet or ICA bases. More specifically, assume given a sequence of measurements $(x^t)_{t=0}^T$ with observation model $x^t = \mathbf{A}s^t + v^t$, for every $0 \leq t \leq T$. Assume that each s^t is drawn independently from a distribution $p_s(s)$, each v^t is drawn independently from a distribution $p_v(v)$, and the prior distribution $p_{\mathbf{A}}(\mathbf{A})$ of **A**. Then the posterior distribution of $(\mathbf{A}, (s^t)_{t=0}^T)$ given $(x^t)_{t=0}^T$ is given by

$$p\left(\mathbf{A}, (s^t)_{t=0}^T | (x^t)_{t=0}^T\right) = \frac{1}{p_x(x)} \prod_{t=0}^T p_v(x^t - \mathbf{A}s^t) p_s(s^t) p_{\mathbf{A}}(\mathbf{A}). \tag{19}$$

The MAP estimator of $(\mathbf{A}, (s^t)_{t=0}^T)$ is obtained my maximizing the above probability. Typically optimization algorithms (Zibulevski and Pearlmutter, 2001) iterate between optimization over **A** for fixed $(s^t)_{t=0}^T$, and optimization over $(s^t)_{t=0}^T$ for fixed **A**. In this work, we concentrate on the latter optimization problem that is given **A**, we look for the MAP estimator of $(s^t)_{t=0}^T$.

III. MAIN RESULTS

Consider the following two optimization problems

$$s^0(\mathbf{x}) = \arg\min_{\mathbf{s} \in \mathbf{C}^n} \|\mathbf{x} - \mathbf{A}\mathbf{s}\|^2 + \mu\|\mathbf{s}\|_0, \tag{20}$$

$$s^1(\mathbf{x}) = \arg\min_{\mathbf{s} \in \mathbf{C}^n} \|\mathbf{x} - \mathbf{B}\mathbf{s}\|^2 + \lambda\|\mathbf{s}\|_1, \tag{21}$$

where $\mathbf{A}, \mathbf{B} \in \mathbf{C}^{n \times n}$, $\mu, \lambda > 0$ and $\mathbf{x} \in \mathbf{C}^n$ are given. In general, the optimizer in (20) may not be unique, in which case $s^0(\mathbf{x})$ denotes one such optimizer. On the other hand, since the criterion (21) is strictly convex, the optimizer in (21) is unique, and $s^1(\mathbf{x})$ is a well-defined function.

The main results of this paper are stated as follows:

Proposition 1. *Consider the optimization problem (21) with* **B** *invertible and* $\lambda > 0$. *Then for every subset* $\mathbf{I} \subset \{1, 2, \ldots, n\}$, *there is a set* $E_{\mathbf{I}} \subset \mathbf{C}^n$ *with nonempty interior so that for* $\mathbf{x} \in E_{\mathbf{I}}$, supp$(s^1(\mathbf{x}))$ **I**.

Proposition 2. *Consider the optimization problem (20) with* **A** *invertible and* $\mu > 0$. *Then for every subset* $\mathbf{I} \subset \{1, 2, \ldots, n\}$, *there is a set* $F_{\mathbf{I}} \subset \mathbf{C}^n$ *with nonempty interior so that for* $\mathbf{x} \in F_{\mathbf{I}}$ *there is an optimizer* $s^0(\mathbf{x})$ *such that* supp$(s^0(\mathbf{x})) = \mathbf{I}$.

Theorem 3. *Assume* **A** *is invertible and* $\mu > 0$ *is given constant. Then there are* $\alpha = \alpha(\mathbf{A}) > 0$ *and sets with nonempty interiors* $\mathbf{D}_{\mathbf{I}}$ *indexed by subsets* $\mathbf{I} \subset \{1, 2, \ldots, n\}$ *such that for* $\mathbf{B} = (\mathbf{A}^{-1})^*$ *(the adjoint of the inverse matrix),* $\lambda = \lambda = \sqrt{8\mu/\alpha}$, *and for every subset* $\mathbf{I} \subset \{1, 2, \ldots, n\}$, $\mathbf{x} \in \mathbf{D}_{\mathbf{I}}$, *and at least one optimizer* $s^0(\mathbf{x})$ *has*

$$\text{supp}(s^0(\mathbf{x})) = \text{supp}(s^1(\mathbf{x})) = \mathbf{I}. \tag{22}$$

Remark 4. With the notations above, the main result simply says $E_{\mathbf{I}} \cap F_{\mathbf{I}}$ has nonempty interior for every $\mathbf{I} \subset \{1, 2, \ldots, n\}$, when $\mathbf{B} = (\mathbf{A}^{-1})^*$ and $\lambda = \lambda = \sqrt{8\mu/\alpha}$.

The function $\alpha = \alpha(\mathbf{A})$ has an explicit description that we present next. First a lemma:

Lemma 5. *Assume* $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ *is a set of independent vectors in* $\mathbf{C}^n$. *Consider two sets* $\mathbf{I}, \mathbf{J} \subset \{1, 2, \ldots, n\}$ *so that* $|\mathbf{I}| \geq |\mathbf{J}|$. *Denote by* **P** *the orthogonal projection onto the span of* $\{\mathbf{v}_j; j \in \mathbf{J}\}$. *Then the set* $\{(1-\mathbf{P})\mathbf{v}_i; i \in \mathbf{I} \setminus \mathbf{J}\}$ *is independent in* $\mathbf{C}^n$.

This lemma says that for every two sets $\mathbf{I}, \mathbf{J} \subset \{1, 2, \ldots, n\}$ with $|\mathbf{I}| \geq |\mathbf{J}|$, and denoting by $\mathbf{A}_i$ the ith column of **A**, and by **P** the orthogonal projection onto the span of $\{A_j; j \in \mathbf{J}\}$ *the set* $\{(1-\mathbf{P})A_i; i \subset \mathbf{I}\backslash\mathbf{J}\}$ is a Riesz basis for its span (Daubechies, 2003), hence there is a $a(\mathbf{I}, \mathbf{J}) > 0$ *so that*:

$$\left\| \sum_{i\in\mathbf{I}\backslash\mathbf{J}} c_i(1-\mathbf{P})A_i \right\|^2 \geq a(\mathbf{I},\mathbf{J}) \sum_{i\in\mathbf{I}\backslash\mathbf{J}} |c_i|^2 \quad \forall c_1,\ldots,c_n \in \mathbf{C}. \quad (23)$$

Then define α as the minimum of Riesz basis lower bounds $a(\mathbf{I}, \mathbf{J})$ over all pairs of subsets $(\mathbf{I}, \mathbf{J})$ of $\{1, 2, \ldots, n\}$ with $|\mathbf{I}| \geq |\mathbf{J}|$

$$\alpha = \min_{\mathbf{I},\mathbf{J}:|\mathbf{I}|\geq|\mathbf{J}|} a(\mathbf{I},\mathbf{J}). \quad (24)$$

IV. PROOF OF RESULTS

Let us start by proving first Lemma 5.

Proof of Lemma 5: Consider $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ independent vectors in $\mathbf{C}^n$, and $\mathbf{I}, \mathbf{J} \subset \{1, 2, \ldots, n\}$ with $|\mathbf{I}| \geq |\mathbf{J}|$. Denote by $\mathbf{P}$ the orthogonal projection onto span $\{\mathbf{v}_j; j \in \mathbf{J}\}$ and by $\mathbf{Q} = 1-\mathbf{P}$, the projection onto its orthogonal complement. We need to prove $\{\mathbf{Q}\mathbf{v}_i; i \in \mathbf{I}\backslash\mathbf{J}\}$ is independent. Assume this is not so. Then there are $c_i \in \mathbf{C}$, $i \in \mathbf{I}\backslash\mathbf{J}$, not all zero so that

$$\sum_{i\in\mathbf{I}\backslash\mathbf{J}} c_i\mathbf{Q}\mathbf{v}_i = 0.$$

Hence

$$\sum_{i\in\mathbf{I}\backslash\mathbf{J}} c_i\mathbf{v}_i \in \text{span}\{\mathbf{v}_j;\ j \in \mathbf{J}\}.$$

But then, there should exist $d_j \in \mathbf{C}$, $j \in \mathbf{J}$ so that

$$\sum_{i\in\mathbf{I}\backslash\mathbf{J}} c_i\mathbf{v}_i = \sum_{j\in\mathbf{J}} d_j\mathbf{v}_j.$$

Thus we obtained a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$, with not all coefficients zero, which is zero. Contradiction with the independence hypothesis. □

Now we prove Theorem 3. A sketch of its proof is as follows. First we construct explicit solutions of (21) and prove Proposition 2. Then we show that there is some $\mathbf{x} \in E_\mathbf{I}$ that is also in $F_\mathbf{I}$, and furthermore it is an interior point in both that concludes the proof of Theorem 3.

Consider the l^1-optimization problem (21).

Definition 6. We call a pair $(\mathbf{s}, \mathbf{x}) \in \mathbf{C}^n \times \mathbf{C}^n$ *admissible* if it satisfies the following set of conditions:

$$(\mathbf{B}*\mathbf{Bs}-\mathbf{B}*\mathbf{x})_k + \frac{\lambda}{2}\frac{s_k}{|s_k|} = 0 \quad \text{for all } k \text{ so that } s_k \neq 0, \quad (25)$$

$$|(\mathbf{B}*\mathbf{Bs}-\mathbf{B}*\mathbf{x})_j| \leq \frac{\lambda}{2} \quad \text{for all } j \text{ so that } s_j = 0. \quad (26)$$

Then we have the following Lemma:

Lemma 7. $\mathbf{s}$ *is a solution of the optimization problem* (21) *if and only if* $(\mathbf{s}, \mathbf{x})$ *is an admissible pair.*

In other words, this lemma says that $(s^1(\mathbf{x}),\mathbf{x})$ *satisfies (25) and (26) and conversely, any solution of (25) and (26) is an optimizer.*

Proof:

First note that the criterion

$$J_1(\mathbf{s}) = \|\mathbf{x}-\mathbf{Bs}\|^2 + \lambda\|\mathbf{s}\|_1, \quad (27)$$

is strictly convex, and therefore it has a unique global minimum.

"$\Rightarrow$" Assume $\mathbf{s} = s^1(\mathbf{x})$ is the unique minimum. Denote by $\mathbf{I}$ the index set of nonzero entries of $\mathbf{s}$, that is $\mathbf{I} = \{k;\ s_k \neq 0\}$, and denote by $\mathbf{e}_k$ the kth vector of the canonical basis, i.e., all entries are zeros except for one "1" on the kth position. Since for $k \in \mathbf{I}$, $\mathbf{J}_{I,k}(t) = J_1(\mathbf{s} + t\mathbf{e}_k)$ has a minimum and is differentiable at $t = 0$, $\partial J_1/\partial s_k|_{\mathbf{s}=s^1}(\mathbf{x}) = 0$. The partial derivative is exactly the left-hand-side of (25). For $j \in \mathbf{I}$, the situation is different. We compute the variation $J_1(\mathbf{s} + t\mathbf{e}_j) - J_1(\mathbf{s})$. Expanding the quadratic form we obtain

$$J_1(\mathbf{s}+t\mathbf{e}_j) - J_1(\mathbf{s}) = t^2\|B_j\|^2 - t\langle B_j, \mathbf{x}-\mathbf{Bs}\rangle - t\langle \mathbf{x}-\mathbf{Bs}, B_j\rangle - \lambda|t|,$$

where B_j is the jth column of $\mathbf{B}$. Choosing $t = \varepsilon\ \langle \mathbf{x}-\mathbf{Bs}, B_j\rangle = \varepsilon\ (\mathbf{B}^*(\mathbf{x}-\mathbf{Bs}))_j$, with $\varepsilon > 0$ arbitrary small, we obtain:

$$J_1(\mathbf{s}+t\mathbf{e}_j) - J_1(\mathbf{s}) = \mathrm{O}(\varepsilon^2) + |\varepsilon||\mathbf{B}^*(\mathbf{x}-\mathbf{Bs}))_j|(\lambda - 2|(\mathbf{B}^*(\mathbf{Bs}-\mathbf{x}))_j|).$$

In order for this to be positive for all $\varepsilon > 0$, the last term should always be nonnegative, meaning

$$\lambda - 2|(\mathbf{B}^*(\mathbf{Bs}-\mathbf{x}))_j| \geq 0,$$

and thus (26).

"$\Leftarrow$" Assume now that $(\mathbf{s}, \mathbf{x})$ is an admissible pair. Then compute the variation $J_1(\mathbf{s} + \mathbf{v}) - J_1(\mathbf{s})$, where $\mathrm{w} = \sum_k w_k\mathbf{e}_k$ is an arbitrary vector with $|w_k| < |s_k|$ for $k \in \mathbf{I}$. We obtain:

$$J_1(\mathbf{s}+\mathbf{w}) - J_1(\mathbf{s}) = \|\mathbf{Bw}\|^2 + \sum_{k\in\mathbf{I}} [2\mathrm{Re}((\mathbf{B}^*\mathbf{Bs}-\mathbf{B}^*\mathbf{x})_k\bar{w}_k) + \lambda(|s_k+w_k| - |s_k|)] + \sum_{k\notin\mathbf{I}} [2\mathrm{Re}((\mathbf{B}^*\mathbf{Bs}-\mathbf{B}^*\mathbf{x})_k\bar{w}_k) + \lambda|w_k|].$$

Now use (25) in the first sum over $k \in \mathbf{I}$, and then (26) to obtain the inequality below

$$J_1(\mathbf{s}+\mathbf{w}) - J_1(\mathbf{s}) = \|\mathbf{Bw}\|^2 + \sum_{k\in\mathbf{I}} \lambda\left[|s_k+w_k| - |s_k| - \mathrm{Re}\left(\frac{s_k\bar{w}_k}{|s_k|}\right)\right] + \sum_{j\notin\mathbf{I}} 2|w_j|\left[\frac{\lambda}{2} + \mathrm{Re}\left((\mathbf{B}^*\mathbf{Bs}-\mathbf{B}^*\mathbf{x})_j\frac{\bar{w}_j}{|w_j|}\right)\right] \geq \sum_{k\in\mathbf{I}} \lambda\left[|s_k+w_k| - |s_k| - \mathrm{Re}\left(\frac{s_k\bar{w}_k}{|s_k|}\right)\right].$$

Now a little algebra shows

$$|s_k+w_k| - |s_k| - \mathrm{Re}\left(\frac{s_k\bar{w}_k}{|s_k|}\right) = \frac{1}{2|s_k|}[|s_k| + |w_k| - |s_k+w_k|][|s_k+w_k| + |w_k| - |s_k|],$$

which is always positive by application of the triangle inequality twice, once in each term. Therefore, if $(\mathbf{s}, \mathbf{x})$ is an admissible pair, $J_1(\mathbf{s} +$

w)$-J_1(\mathbf{s}) \geq 0$ is a neighborhood of zero. Hence s is a local minimum for $J_1(\cdot)$, but since the local minimum is also global, $\mathbf{s} = s^1(\mathbf{x})$. □

Next we construct particular admissible pairs. Fix $\mathbf{I}$ an arbitrary subset of $\{1, 2, \ldots, n\}$, possibly the empty set. We will construct an admissible pair $(\hat{\mathbf{s}}, \hat{\mathbf{x}})$ so that $\text{supp}(\hat{\mathbf{s}}) = \mathbf{I}$. Consider a $\hat{\mathbf{s}} \in \mathbf{C}^n$ so that $\text{supp}(\hat{\mathbf{s}}) = \mathbf{I}$ and $\|\hat{\mathbf{s}}\| < \lambda/4\,\|\mathbf{A}\|^2$. Then define for $k \in I$

$$\xi_k = \sum_{l \in \mathbf{I}} (\mathbf{B}^*\mathbf{B})_{k,l} S_l + \frac{\lambda}{2} \frac{\hat{s}_k}{|s_k|}, \tag{28}$$

and construct

$$\hat{\mathbf{x}} = \sum_{k \in \mathbf{I}} \xi_k (\mathbf{B}^{-1})^* \mathbf{e}_k. \tag{29}$$

We claim $(\hat{\mathbf{s}}, \hat{\mathbf{x}})$ is an admissible pair. Indeed, first note $(\mathbf{B}^* \hat{\mathbf{x}})_k = \xi_k$ for $k \in \mathbf{I}$, and $(\mathbf{B}^* \hat{\mathbf{x}})_j = 0$ for $j \notin \mathbf{I}$. Then (28) proves (25), whereas (26) is satisfied by the norm constraint $\|\hat{\mathbf{s}}\| < \lambda/4\,\|\mathbf{A}\|^2$.

The third step is to show that for an admissible pair $(\hat{\mathbf{s}}, \hat{\mathbf{x}})$, as constructed above, there is an open neighborhood of $\hat{\mathbf{x}}$, say $E_{\mathbf{I}}$, so that for every $\mathbf{x} \in E_{\mathbf{I}}$ the optimal solution $s^1(\mathbf{x})$ has support $\mathbf{I}$

$$\text{supp}(s^1(\mathbf{x})) = \mathbf{I} \quad \forall\, \mathbf{x} \in E_{\mathbf{I}}.$$

Note first that at $(\hat{\mathbf{s}}, \hat{\mathbf{x}})$, (26) is actually satisfied as

$$|(\mathbf{B}^*\mathbf{B}\hat{\mathbf{s}} - \mathbf{B}^*\hat{\mathbf{x}})_j| < \frac{\lambda}{4} < \frac{\lambda}{2}.$$

By continuity of the left-hand-side there are $r_1, r_2 > 0$ so that for every $\mathbf{s}, \mathbf{x} \in \mathbf{C}^n$, $\|\mathbf{s}-\hat{\mathbf{s}}\| < r_1$, $\|\mathbf{x}-\hat{\mathbf{x}}\| < r_2$,

$$|(\mathbf{B}^*\mathbf{B}\mathbf{s} - \mathbf{B}^*)_j| < \frac{\lambda}{2}.$$

Now set

$$F_j^k\colon\ l*(\mathbf{I}) \times l*(\mathbf{I}) \times \mathbf{C}^n \to \mathbf{C} \quad k \in \mathbf{I} \quad j \in \{1,2\},$$

$$F_1^k(\mathbf{u}, \mathbf{v}, \mathbf{x}) = \sum_{l \in \mathbf{I}} (\mathbf{B}^*\mathbf{B})_{k,l} S_l + \frac{\lambda}{2} \frac{u_k}{\sqrt{u_k \mathbf{v}_k}} - (\mathbf{B}^*\mathbf{x})_k,$$

$$F_2^k(\mathbf{u}, \mathbf{v}, \mathbf{x}) = \sum_{l \in \mathbf{I}} \overline{(\mathbf{B}^*\mathbf{B})_{k,l}} S_l + \frac{\lambda}{2} \frac{\mathbf{v}_k}{\sqrt{u_k \mathbf{v}_k}} - \overline{(\mathbf{B}^*\mathbf{x})_k},$$

where $l^*(\mathbf{I})$ is the set of $\mathbf{I}$-indexed vectors with nonvanishing components. An admissible pair $(\mathbf{s}, \mathbf{x})$ with $\text{supp}(\mathbf{s}) = \mathbf{I}$ satisfies $F_j^k(\mathbf{s}, \bar{\mathbf{s}}, \mathbf{x}) = 0$, for all $k \in \mathbf{I}$ and $j \in \{1, 2\}$. Our task is to show tha t there is $\mathbf{u} = u(\mathbf{x})$ so that $F_j^k(u(\mathbf{x}), \overline{u(\mathbf{x})}, \mathbf{x}) = 0$ for all $k \in \mathbf{I}, j \in \{1, 2\}$ and $\mathbf{x}$ in a neighborhood of $\hat{\mathbf{x}}$. This follows from the Implicit Mapping Theorem, provided the Jacobian of $(F_j^k)_{k\in\mathbf{I}, j\in\{1,2\}}$ with respect to $(\mathbf{u}, \mathbf{v})$ is nonzero at $(\pi(\hat{\mathbf{s}}), \pi(\bar{\hat{\mathbf{s}}}), \hat{\mathbf{x}})$ (where $\pi : l^*(\{1, 2, \ldots, n\}) \to l^*(\mathbf{I})$ is the reduction, or cut-off, operator to index set $\mathbf{I}$). The Jacobian at $(\pi(\hat{\mathbf{s}}), \pi(\bar{\hat{\mathbf{s}}}), \hat{\mathbf{x}})$ turns out to be the determinant of

$$\mathbf{J} = \begin{bmatrix} \pi\mathbf{B}^*\mathbf{B}_\pi^* + \frac{\lambda}{2}\text{Diag}\left(\frac{1}{|\hat{s}_k|}\right) & -\frac{\lambda}{4}\text{Diag}\left(\frac{\hat{s}_k}{\bar{\hat{s}}_k|\hat{s}_k|}\right) \\ -\frac{\lambda}{4}\text{Diag}\left(\frac{\bar{\hat{s}}_k}{\hat{s}_k|\hat{s}_k|}\right) & \pi\overline{\mathbf{B}^*\mathbf{B}}_\pi^* + \frac{\lambda}{4}\text{Diag}\left(\frac{1}{|\hat{s}_k|}\right) \end{bmatrix}.$$

We compute the quadratic form $(x_1^*, x_2^*)\mathbf{J}(x_1; x_2)^{\mathrm{T}}$ with $x_1, x_2 \in l^2(\mathbf{I})$

$$(x_1^*, x_2^*)\mathbf{J}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \|\mathbf{B}\pi^* x_1\|^2 + \|\mathbf{B}\pi^* \bar{\mathbf{x}}_2\|^2 + \frac{\lambda}{4}\sum_{k \in \mathbf{I}} \frac{|x_{1,k} - x_{2,k} e^{-2\bar{i}\varphi_k}|^2}{|\hat{s}_k|} \geq \lambda_{\min}(\mathbf{B}^*\mathbf{B})\left(\|x_1\|^2 + \|x_2\|^2\right),$$

where $e^{-\bar{i}\varphi k} = \hat{s}_k/|\hat{s}_k|$. Hence $\mathbf{J}$ is invertible and $\det(\mathbf{J}) > 0$. By the inverse mapping theorem, we thus have obtained a neighborhood of $\hat{\mathbf{x}}$ where $s^1(\mathbf{x})$ always has support exactly $\mathbf{I}$. This concludes the proof of Proposition 1. □

Let us turn now our attention to Proposition 2. For every $\mathbf{I} \in \{1, 2, \ldots, n\}$ we construct a solution of (20), which is stable under perturbations in $\mathbf{x}$. This will prove the result.

Fix $\mathbf{I} \subset \{1, 2, \ldots, n\}$. Set

$$\mathbf{x}^0 = \sum_{k \in \mathbf{I}} \xi_k^0 \mathbf{A}\mathbf{e}_k \quad \xi_k^0 = \sqrt{\frac{2\mu}{\alpha}}\mathbf{e}^{\bar{i}\varphi_k}, \tag{30}$$

where $(\varphi_k)_{k\in\mathbf{I}}$ are some arbitrary phases. Define

$$\mathbf{s}^0 = \sum_{k \in \mathbf{I}} \xi_k^0 \mathbf{e}_k. \tag{31}$$

We claim $\mathbf{s}^0$ above is a solution of (20) for $\mathbf{x} = \mathbf{x}^0$. Denote

$$J_0(\mathbf{s}) = \|\mathbf{x}^0 - \mathbf{A}\mathbf{s}\|^2 + \mu\|\mathbf{s}\|_0. \tag{32}$$

We need to show $J_0(\mathbf{s}^0) \leq J_0(\mathbf{s})$, for all $\mathbf{s} \neq \mathbf{s}^0$. Note first $J_0(\mathbf{s}^0) = \mu\,|\mathbf{I}|$.

Let $\mathbf{s} \in \mathbf{C}^n$ be arbitrary. If $|\text{supp}(\mathbf{s})| > |\mathbf{I}|$, then clearly $J_0(\mathbf{s}) \geq \mu\,|\text{supp}(\mathbf{s})| > \mu\,|\mathbf{I}| = J_0(\mathbf{s}^0)$.

Assume now $|\text{supp}(\mathbf{s})| \leq |\mathbf{I}|$. Let us denote $\mathbf{J} = \text{supp}(\mathbf{s})$. Notice that $\mathbf{x}^0 \notin \text{span}\{\mathbf{A}\mathbf{e}_j;\ j \in \mathbf{J}\}$ because otherwise this would imply the columns of $\mathbf{A}$ are not independent. Let us denote by $\mathbf{P}$ the orthogonal projection onto the span of $\{\mathbf{A}\mathbf{e}_j; j \in \mathbf{J}\}$. Then:

$$J_0(\mathbf{s}) \geq \|(1-\mathbf{P})\mathbf{x}^0\|^2 + |\mathbf{J}|\mu = \left\|\sum_{k \in \mathbf{I}\setminus\mathbf{J}} \sqrt{\frac{2\mu}{\alpha}} e^{i\varphi_k}(1-\mathbf{P})\mathbf{A}\mathbf{e}_k\right\|^2 + |\mathbf{J}|\mu.$$

Now we apply (23) with (24), and obtain

$$J_0(\mathbf{s}) \geq a(\mathbf{I}, \mathbf{J}) \sum_{k \in \mathbf{I}\setminus\mathbf{J}} \frac{2\mu}{\alpha} + |\mathbf{J}|\mu \geq 2\mu(|\mathbf{I}\setminus\mathbf{J}|) + \mu|\mathbf{J}| \geq |\mathbf{I}|\mu + (|\mathbf{I}| - |\mathbf{J}|)\mu > \mu|\mathbf{I}| = J_0(\mathbf{s}^0).$$

This shows that $\mathbf{s}^0$ is the optimum for $J_0(\cdot)$. Furthermore, we also obtain the following inequalities:

$$J_0(\mathbf{s}) - J_0(\mathbf{s}^0) \geq \mu\big|\|\mathbf{s}\|_0 - \|\mathbf{s}^0\|_0\big|,$$

for all $\mathbf{s} \in \mathbf{C}^n$, and

$$J_0(\mathbf{s}) - J_0(\mathbf{s}^0) \geq 2\mu|\mathbf{I}\setminus\mathbf{J}|,$$

when $\|\mathbf{s}\|_0 = |\mathbf{I}|$. Explicitly this means

$$\|\mathbf{x}^0 - \mathbf{A}\mathbf{s}\|^2 + \mu\|\mathbf{s}\|_0 - \|\mathbf{x}^0 - \mathbf{A}\mathbf{s}^0\|^2 - \mu\|\mathbf{s}^0\|_0 \geq \mu \cdot \max(|\|\mathbf{s}\|_0 - \|\mathbf{s}^0\|_0|, 2|\mathrm{supp}(\mathbf{s})\backslash\mathrm{supp}(\mathbf{s}^0)|). \quad (33)$$

In particular, these show s^0 is the unique solution of (20). The left-hand-side of (33) is a differentiable function on x^0, with Lipschitz constant $L = 2\,\|\mathbf{A}(\mathbf{s}-\mathbf{s}^0)\|$. Therefore, for any $\mathbf{s} \in \mathbf{C}^n$ with $\|\mathbf{s}-\mathbf{s}^0\| < 2/\sigma_{\min}(\mathbf{A})\sqrt{2\mu|\mathbf{I}|} =: S$ with $\sigma_{\min}(\mathbf{A})$ the smallest singular eigenvalue of $\mathbf{A}$, and $\mathrm{supp}(\mathbf{s}) \neq \mathrm{supp}(\mathbf{s}^0)$, and $\mathbf{x} \in \mathbf{C}^n$ with

$$\|\mathbf{x} - \mathbf{x}^0\| < \min(\mu\backslash(2\|\mathbf{A}\|S), \sqrt{\mu|\mathbf{I}|}. \quad (34)$$

we obtain

$$J_0(\mathbf{s},\mathbf{x}) - J_0(\mathbf{s}^0,\mathbf{x}) \leq \mu\max(|\|\mathbf{s}\|_0 - \|\mathbf{s}^0\|_0| - 1, |\mathrm{supp}(\mathbf{s})\backslash\mathbf{I}| - 1) \geq 0,$$

where

$$J_0(\mathbf{s},\mathbf{x}) = \|\mathbf{x} - \mathbf{A}\mathbf{s}\|^2 + \mu\|\mathbf{s}\|_0.$$

Furthermore, for $\|\mathbf{s}-\mathbf{s}^0\| > 2/\sigma_{\min}(\mathrm{A})\sqrt{2\mu|I|}$,

$$2\sqrt{2\mu|\mathbf{I}|} < \sigma_{\min}(\mathbf{A})\|\mathbf{s} - \mathbf{s}^0\| \leq \|\mathbf{A}(\mathbf{s} - \mathbf{s}^0)\| \leq \|\mathbf{x} - \mathbf{A}\mathbf{s}\| + \|\mathbf{x} - \mathbf{A}\mathbf{s}^0\| \leq \|\mathbf{x} - \mathbf{A}\mathbf{s}\| + \sqrt{\mu|\mathbf{I}|}.$$

Hence

$$\mathbf{J}(\mathbf{s},\mathbf{x}) \geq (9 - 4\sqrt{2})\mu|\mathbf{I}| + \mu|\mathrm{supp}(\mathbf{s})| \geq (9 - 4\sqrt{2})\mu|\mathbf{I}|.$$

On the other hand

$$\mathbf{J}(\mathbf{s}^0,\mathbf{x}) = \|\mathbf{x} - \mathbf{x}^0\|^2 + \mu|\mathbf{I}| \leq 2\mu|\mathbf{I}| < \mathbf{J}(\mathbf{s},\mathbf{x}).$$

This shows that for such a $\mathbf{x}$, an optimum $s^0(\mathbf{x})$ has to have the same support as s^0, i.e., $\mathrm{supp}(s^0(\mathbf{x})) = \mathbf{I}$. Hence we obtained a neighborhood of $\mathbf{x}^0$, say $F_{\mathbf{I}}$, so that the optimum solution of (20) has support $\mathbf{I}$. This proves Proposition 2. □

Now we are prepared to prove Theorem 3. The above discussion showed the existence of neighborhoods in $\mathbf{C}^n$, denoted $F_{\mathbf{I}}$, $E_{\mathbf{I}}$, where the optimizers of (21), respectively (34) have support exactly $\mathbf{I}$. Theorem 3 is proved by showing $E_{\mathbf{I}} \cap F_{\mathbf{I}} \neq \emptyset$. But, for $\mathbf{B} = (\mathbf{A}^{-1})^*$ and $\lambda = \sqrt{8\mu/\alpha}$, $\mathbf{x}^0$ of (30) is in the closure of both the set of $\hat{\mathbf{x}}$ defined by (28) and of $\hat{\mathbf{x}}$ defined by (29). Hence the two sets $E_{\mathbf{I}}$ and $F_{\mathbf{I}}$ should have a nonempty interior intersection, and this proves the statement of the Theorem. □

V. AN EXAMPLE

In this section, we present an example of optimization in $\mathbf{R}^2$. Consider the case of problem (20) where

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad \mu = 1. \quad (35)$$

This data turns (20) into

$$(s_1^0(x), s_2(x)) = \underset{s_1,s_2}{\arg\min}\left\{|x_1 - s_1 - s_2|^2 + |x_2 - s_2|^2 + 1_{s_1\neq 0} + 1_{s_2\neq 0}\right\}, \quad (36)$$

where $I_{s\neq 0}$ is 1 if $s \neq 0$, and 0 for $s = 0$.

Let us state the l^0 optimization problem. To compute α we need to consider only ($\mathbf{I} = \{1\}$, $\mathbf{J} = \{2\}$) and ($\mathbf{I} = \{2\}$, $\mathbf{J} = \{1\}$), because $\mathbf{I} = \{1, 2\}$ and $\mathbf{J} = \{1\}$ (or $\mathbf{J} = \{2\}$) reduces to one of these two cases. The lower Riesz basic sequence bound is the norm of the projection of the corresponding column vector onto the orthogonal direction to the other column vector. The bound for ($\mathbf{I} = \{1\}$, $\mathbf{J} = \{2\}$) is $a(\mathbf{I}, \mathbf{J}) = ½$, whereas the bound of ($\mathbf{I} = \{2\}$, $\mathbf{J} = \{1\}$) is $a(\mathbf{I},\mathbf{J}) = 1$. Hence $\alpha = ½$ and therefore the associated l^1-optimization problem (21) has the following parameters:

$$B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \quad \lambda = 4. \quad (37)$$

This data turns (21) into

$$(s_1^1(x), s_2^1(x)) = \underset{s_1,s_2}{\arg\min}\left\{|x_1 - s_1|^2 + |x_2 + s_1 - s_2|^2 + 4|s_1| + 4|s_2|\right\}. \quad (38)$$

After a few computations, one can obtain the solutions in closed form as follows. The l^0 optimization problem (36) has the following solution:

$$s_1^0 = 0, s_2^0 = 0 \quad \text{for } (x_1,x_2) \in \{x_1^2 + x_2^2 \leq 2\} \cap \{|x_1| \leq 1\} \cap \{|x_1 + x_2| \leq \sqrt{2}\}.$$

$$s_1^0 = 0, s_2^0 = \frac{x_1 + x_2}{2} \quad \text{for } (x_1,x_2) \in \{|x_1 - x_2| \leq \sqrt{2}\} \cap \{|x_1 + x_2| \geq \sqrt{2}\} \cap \left\{\frac{x_2}{x_1} \leq \sqrt{2} - 1\right\}.$$

$$s_1^0 = x_1, s_2^0 = 0 \quad \text{for } (x_1,x_2) \in \{|x_1| \geq 1\} \cap \{|x_2| \leq 1\} \cap \left\{\frac{x_2}{x_1} \leq \sqrt{2} - 1\right\}.$$

$$s_1^0 = x_1 - x_2, s_2^0 = x_2 \quad \text{for } (x_1,x_2) \in \{|x_2| \geq 1\} \cap \{|x_1 - x_2| \geq 1\} \cap \{x_1^2 + x_2^2 \geq 2\}.$$

Figure 1 shows the data domains $E_{\mathbf{I}}$ in the input space, where the solution to the optimization problem, has some specific support. At the intersection of domains (on the frontiers), the optimizer may be degenerate.

The l^1 optimization problem (38) has the following solution:

$$s_1^1 = 0, s_2^0 = 0 \quad \text{for } (x_1,x_2) \in \{|x_2| \leq 2\} \cap \{||x_2 - x_1| \leq 2\}.$$

$$s_1^1 = 0, s_2^1 = 2\theta\left(\frac{x_2}{2}\right) \quad \text{for } \{|x_2| > 2\} \cap \{|x_1 - 2\,\mathrm{sign}(x_1)| \leq 2\}.$$

$$s_1^1 = \theta\left(\frac{x_1 - x_2}{2}\right), s_2^1 = 0 \quad \text{for } (x_1,x_2) \in \{|x_2 - x_1| > 2\} \cap \{|x_1 + x_2 - 2\,\mathrm{sign}(x_1)| \leq 4\}.$$

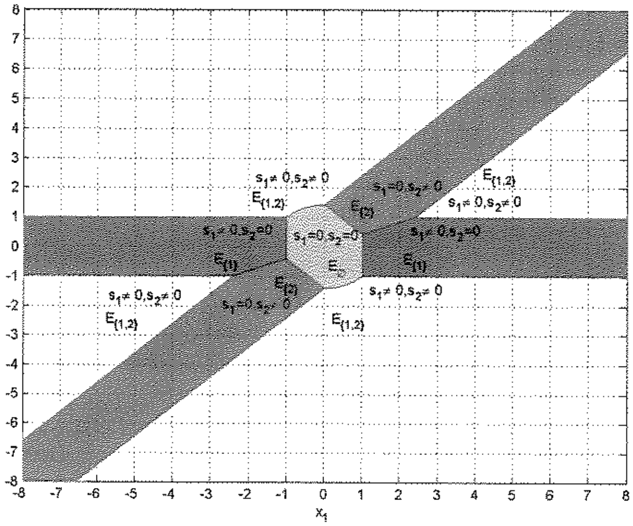

Figure 1. The data domains in (x_1, x_2) space with same support of solution of l^0 optimization (36).

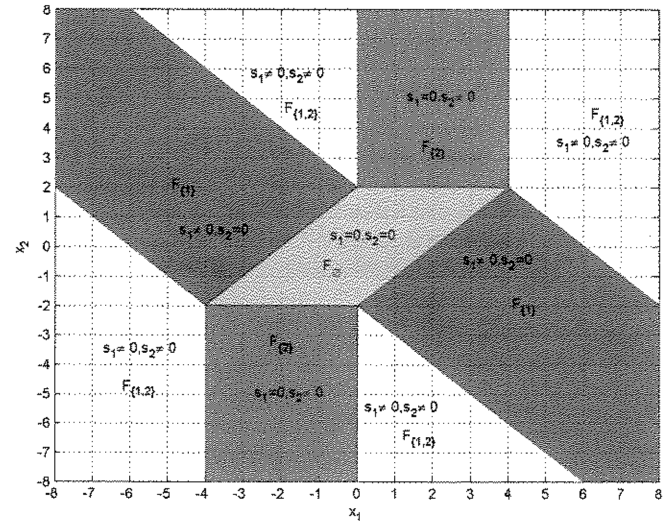

Figure 2. The data domains in (x_1, x_2) space with same support of solution of l^1 optimization (38).

$$s_1^1 = x_1 - 2[\text{sign}(s_1^1) + \text{sign}(s_2^1)],$$
$$s_2^1 = x_1 + x_2 - 2[\text{sign}(s_1^1) + 2\,\text{sign}(s_2^1)] \quad \text{for the rest,}$$

where

$$\theta(x) = \begin{cases} x - 1 & \text{for } x \geq 1, \\ 0 & \text{for } |x| < 1, \\ x + 1 & \text{for } x \leq -1. \end{cases}$$

Figure 2 shows the data domains for the l^1 optimization problem. Within each domain, the solution has the same support.

Overlapping the Figure 1 and Figure 2 we obtain Figure 3. The intersections where supports of the solutions of the two problems coincide, $D_\emptyset$, $D_{\{1\}}$, $D_{\{2\}}$, $D_{\{1,2\}}$, describe the set of input data where the method presented in this paper correctly solves the hard l^0 problem by first solving the easier l^1 problem. The unlabeled shaded portions of the graph correspond to inputs for which the method described in this paper would fail to determine the correct l^0 solution. One of the main results of this paper is that the set of input data for which the supports of the solution to the l^0 problem and appropriately constructed l^1 problem coincide is nonempty, and this fact is clearly verified in Figure 3.

VI. CONCLUSIONS

Estimation of a sparse data or signal representation is hard. We present a new approach to the corresponding optimization problem, which shows that for a nontrivial set of input data the problem is equivalent to and can be solved by an algorithm devised for the simpler low-spread optimization problem. This does not mean a reduction of the harder l^0 problem to a simpler l^1 problem in all cases, however. The two optimization problems are to minimize the Euclidian norm of linear approximation error with an l^0 penalty, or with an l^1 penalty. The latter problem can be solved efficiently by an iterative algorithm. Here, for a given l^0 optimization problem, we construct an associated l^1 optimization problem and show that for a set with open interior of the input data the optimizers of the two optimization problems have the same support. Once the support of the l^0 problem is known, the actual solution is easily found by solving a linear system of equations. Thus the associated l^1 optimization problem is used to find the support of the l^0 optimizer and this leads to the optimal l^0 solution when the two optimization problems have the same support. When the optimizers do not have the same support, the method will fail to produce the optimal solution.

This class of optimization problems is related to a number of signal estimation problems of interest. The MAP estimator of a signal with generalized exponential prior in the presence of Gaussian noise reduces to an optimization problem of the type studied here. Similarly, regularization problems with exponential cost reduce to the same optimization problem.

Our result can be applied to a new class of sparse signal representation techniques, for example speech enhancement techniques, which use signal-adapted representations instead of the standard frequency-domain representation. Such representations use for instance the ICA technique to replace the Fourier transform by a more dense but, hopefully, better signal adapted transformation that represents the signal in a much sparser form.

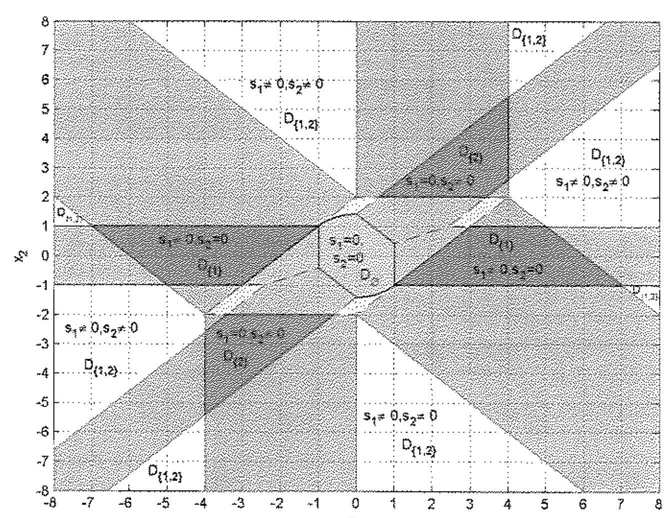

Figure 3. The intersection of the data domains where supports of (36) and (38) coincide.

Several issues remain as topics of further study. One such issue is the ''size'' of regions where the supports of the two optimization problems overlap. We have shown here only that the regions have nonzero size. Another topic concerns the redundant case, namely the case when $\mathbf{A}$ is a $\mathbf{C}^{n \times m}$ matrix with $m > n$, which is not addressed in this work.

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