Multiframes and Multi Riesz Bases: I.The General Theory and Weyl-Heisenberg Case

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Abstract. In this paper the concepts of *multiframe* and *multi Riesz basis* are introduced and studied. The theory is further developed for Weyl-Heisenberg frames and Riesz bases. A multiframe represents a collection of frames having identical index sets and fulfilling some algebraic and topologic conditions. When a similar set of algebraic and topologic conditions are fulfilled by a family of Riesz bases we call this collection a multi Riesz basis. In the Weyl-Heisenberg case a kind of *duality principle* is shown to hold. The multiframe concept is found to be very useful in the multiplexing of signals theory (hence the name) whereras multi Riesz bases are useful in the theory of modulation of signals.

1. Introduction

The concept of *frame* was introduced more than forty years ago by R.J.Duffin and A.C.Schaeffer in their celebrated paper [**DuSch52**]. This concept was taken up again later in the mid-80's by I.Daubechies, A.Grossman and Y.Meyer in [**DGM86**]. Then, after the setting off of wavelet theory at the end of 80's, the frame theory has known an explosion too. The expository survey by C.Heil and D.Walnut ([**HaWa89**]) summarizes the basic properties of Weyl-Heisenberg and wavelet frames proved by that time.

Necessary and sufficient conditions to have a Weyl-Heisenberg or wavelet frame were studied by I.Daubechies in [**Daub90**] and later refined by C.K.Chui and X.Shi in [**ChSh93a**] and [**ChSh93b**].

Weyl-Heisenberg frames and Riesz bases have been thoroughly studied in the last few years; see for instance volume 1, no.4 (1995) of the Journal of Fourier Analysis and Applications where three seminal papers were published ([**BHW95, Jans95, DLL95**]). The present paper extends their results from the 1-frame case to the multiframe and multi Riesz basis setting.

A very important tool in analysis of Weyl-Heisenberg frames is the Zak transform. This transform has been (re)discovered many times and goes by many names (Gauss, Weil-Brezin, Gel'fand, Zak). We name it here after J.Zak who used it systematically in connection with solid state physics and analysis ([**Zak67, Zak72,**

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BGZ75]). In connection with the Gabor decomposition of distributions, the Zak transform was used by A.J.E.M.Janssen in [**Jans81**] and next in [**Jans82, Jans88**]. Later on it has been used by R.Balian to motivate the nonlocalization of Weyl-Heisenberg Riesz bases ([**Bali81**]).

Two extensions of the frames standard theory have been proposed. In the first extension the Hilbert space is replaced by a Banach space and thus one obtains Banach frames (see [**ChHe97**]). The second extension deals with an uncountable index set leading to a continuous frame; this approach is due to S.Ali, J.Antoine and J.Gazeau in [**AAG93**]. The extension presented in this paper is different from either. Our approach is based on the geometric study of a family of (Hilbert) frames indexed by the same (countable) index set.

The structure of the paper is the following: in section 2 we recall basic results on frames and Riesz basis and we introduce the notions of multiframe and multi Riesz basis; in section 3 we study the Weyl-Heisenberg multiframe and multi Riesz basis and we obtain sufficient conditions for these sets to have coherent duals;

2. Multiframes and Multi Riesz bases

2.1. Let H denote a separable Hilbert space and $\mathbf{I}$ a countable index set. Let us denote by $S_0(\mathbf{I})$ the set of all finite supported complex-valued sequences on $\mathbf{I}$ (i.e. $c \in S_0(\mathbf{I})$ if $c(j) \neq 0$ for only a finite number of $j \in \mathbf{I}$).

A family of vectors $\mathcal{F} = \{f_i, i \in \mathbf{I}\}$ is called a *frame* for H if there are two positive constants $0 < A \leq B < \infty$ such that, for all $f \in H$:

$$A\|f\|^2 \leq \sum_{i\in\mathbf{I}} |<f, f_i>|^2 \leq B\|f\|^2 \tag{2.1}$$

The two constants A and B are called the *frame bounds*. When $A = B$ the frame is called *tight*. A *Bessel sequence* (or *set*) is a family of vectors $\mathcal{F}$ for which the upper bound in (2.1) holds. Thus any frame is a Bessel sequence, but the converse is obviously not true.

A family of vectors $\mathcal{R} = \{f_i, i \in \mathbf{I}\}$ is called a *Riesz basis for its span* if there are two positive constants $0 < A \leq B < \infty$ such that, for all $c \in S_0(\mathbf{I})$:

$$A\sum_{i\in\mathbf{I}} |c_i|^2 \leq \|\sum_{i\in\mathbf{I}} c_i f_i\|^2 \leq B\sum_{i\in\mathbf{I}} |c_i|^2 \tag{2.2}$$

When the closure (over $l^2(\mathbf{I})$) of the linear span of $\mathcal{R}$ is H we shall simply call $\mathcal{R}$ a *Riesz basis* (or explicitely, if there is some danger of confusion, a *Riesz basis for H*).

Now we introduce a few operators that will be useful in our geometric description below.

Definition 2.1. Given $\mathcal{F} = \{f_i, i \in \mathbf{I}\}$ which is a frame for H, a Bessel sequence or a Riesz basis for its span, the *analysis operator* T, the *synthesis operator* T^*, the *frame operator* S and the *grammian operator* G are defined as follows:

$$T : H \to l^2(\mathbf{I}) \ , \ T(f) = \{<f, f_i>\}_{i\in\mathbf{I}}$$

$$T^* : l^2(\mathbf{I}) \to H \ , \ T^*(c) = \sum_{i\in\mathbf{I}} c_i f_i$$

$$S : H \to H \ , \ S = T^*T \ , \ S(f) = \sum_{i\in\mathbf{I}} <f, f_i> f_i$$

$$G : l^2(\mathbf{I}) \to l^2(\mathbf{I}) \ , \ G = TT^* \ , \ G(c) = \{< \sum_{i\in\mathbf{I}} c_i f_i, f_j > \}_{j\in\mathbf{I}}$$

Let us denote $RanT$ by E and call it the *coefficient range* of $\mathcal{F}$. We denote $RanT^*$ by R and we call it the *span* of $\mathcal{F}$; note that R always contains the finite linear span of $\mathcal{F}$, but need not be closed if $\mathcal{F}$ is only a Bessel sequence. Both are closed spaces when $\mathcal{F}$ is a frame or a Riesz basis for its span. Moreover, when $\mathcal{F}$ is a frame $R = H$ and S is invertible, whereas when $\mathcal{F}$ is a Riesz basis for its span, R equals this closed span (hence the name), $E = l^2(\mathbf{I})$ and G is invertible. Note that when $\mathcal{F}$ is a Riesz basis for its span, it is immediately frame for R.

This terminology is borrowed from [**RoSh95**].

2.2. Suppose now $\mathcal{F}$ is a frame for H. A frame $\mathcal{F}' = \{f_i^{'}, i \in \mathbf{I}\}$ for H is called a *dual* of $\mathcal{F}$ if its synthesis operator $T^{'*} : l^2(\mathbf{I}) \to H$, $T^{'*}(c) = \sum_{i\in\mathbf{I}} c_i f_i^{'}$ is a left inverse of T (the analysis operator associated to $\mathcal{F}$). Formally: $T^{'*}T = 1_H$ (the identity of H) and also (by taking the adjoint): $T^*T' = 1_H$; explicitely we obtain the following resolutions of identity:

$$\sum_{i\in\mathbf{I}} < f, f_i > f_i^{'} = \sum_{i\in\mathbf{I}} < f, f_i^{'} > f_i = f \ , \ \forall f \in H \tag{2.3}$$

Note that, in general, the dual frame is not unique. However, it is unique if and only if $\mathcal{F}$ is a Riesz basis for H. There is a distigueshed dual $\tilde{\mathcal{F}} = \{\tilde{f}_i, i \in \mathbf{I}\}$, called the *standard dual*, which is obtained as:

$$\tilde{f}_i = S^{-1} f_i \tag{2.4}$$

A geometric interpretation of the standard dual is the following: it is the only dual frame which has the same coefficient range as the initial frame ([**Li95**]).

Another very useful object linked to the frame $\mathcal{F}$ is the *associated tight frame* $\mathcal{F}^{\#} = \{f_i^{\#}, i \in \mathbf{I}\}$ defined by the following relation:

$$f_i^{\#} = S^{-1/2} f_i \ , \ i \in \mathbf{I} \tag{2.5}$$

It tourns out that $\mathcal{F}^{\#}$ is a tight frame with bound 1 having the same coefficient range as $\mathcal{F}$. Moreover, if E is the coefficient range of $\mathcal{F}$ and P_E denotes the orthogonal projection onto E, then:

$$P_E = T^{\#}(T^{\#})^* = TT^{'*} = T'T^* = TS^{-1}T^* \tag{2.6}$$

where $T^{\#}$ is the analysis operator associated to $\mathcal{F}^{\#}$ and is an isometry between H and $E \subset l^2(\mathbf{I})$. Thus the image of $\mathcal{F}^{\#}$ under $T^{\#}$ is the orthonormal projection onto E of the canonical basis of $l^2(\mathbf{I})$, i.e. $T^{\#} f_i^{\#} = P_E \delta_i$, where $\delta_i = \{\delta_{ij}\}_{j\in\mathbf{I}}$ and δ_{ij} is 1 if $i = j$ and 0 otherwise (the Kronecker symbol). This observation implies the following result ([**Holu94**]):

THEOREM 2.2. *Given an infinite dimensional closed subspace E of $l^2(\mathbf{I})$ and an infinite dimensional separable Hilbert space H, there is a frame $\mathcal{F}$ in H such that its coefficient range is E.*

This geometric result will be useful in the sequel.

Consider now two frames $\mathcal{F}_1 = \{f_i^1, i \in \mathbf{I}\}$ and $\mathcal{F}_2 = \{f_i^2, i \in \mathbf{I}\}$ in H and let E_1, E_2 be their respective coefficient ranges in $l^2(\mathbf{I})$. We define two concepts related to the geometry of these frames:

DEFINITION 2.3. The two frames are called *orthogonal* (and we denote this by $\mathcal{F}_1 \perp \mathcal{F}_2$) if E_1 is orthogonal to E_2 in $l^2(\mathbf{I})$.

DEFINITION 2.4. The pair $(\mathcal{F}_1, \mathcal{F}_2)$ is called a *multiframe* for H if $E_1 \cap E_2 = \{0\}$ and the algebraic direct sum $E_1 \oplus E_2$ is closed.

We note here the following remarks:

REMARK 2.5. The definition of orthogonality can be extended to arbitrary Bessel sequences, not necessarily frames. We say $\mathcal{F}_1$ *is orthogonal to* $\mathcal{F}_2$ *in* $l^2(\mathbf{I})$ *sense*, if $Ran\, T_1$ is orthogonal to $Ran\, T_2$ as linear subspaces of $l^2(\mathbf{I})$.

REMARK 2.6. Suppose E_1 and E_2 are two closed subspaces in the same Hilbert space and $E_1 \cap E_2 = \{0\}$. Let us denote by:

$$\alpha = \inf_{\substack{0 \neq x_1 \in E_1 \\ 0 \neq x_2 \in E_2}} arccos \frac{|< x_1, x_2 >|}{\|x_1\| \cdot \|x_2\|}$$

This is the minimum angle between the two subspaces. Then, the condition that $E_1 \oplus E_2$ is closed is equivalent with $\alpha > 0$.

REMARK 2.7. If T_1, T_2 are the analysis operators associated to the frames $\mathcal{F}_1$ and $\mathcal{F}_2$ respectively, then the orthogonality condition $\mathcal{F}_1 \perp \mathcal{F}_2$ is equivalent to $T_1^* T_2 = 0$

REMARK 2.8. Suppose $\mathcal{F}$ is a frame whose coefficient range has codimension infinity. Then, by Theorem 2.2 there is a frame orthogonal to $\mathcal{F}$. Therefore, the existence of an orthogonal frame is quite trivial. The difficult task will be to find *coherent* orthogonal frames (see below).

REMARK 2.9. If $\mathcal{F}_1$ and $\mathcal{F}_2$ are orthogonal, then obviously $(\mathcal{F}_1, \mathcal{F}_2)$ is a multiframe. Thus the multiframe notion extends the orthogonality condition.

The definition of multiframes can be extended to an arbitrary number of frames.

DEFINITION 2.10. Let $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k$ be k frames in H with the coefficients ranges $E_1, E_2, \dots, E_k$, respectively. Then, the k-tuple $(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k)$ is called a *multiframe* for H if $E_l \cap E_s = \{0\}$, for any $1 \leq l \neq s \leq k$ and the algebraic direct sum $E_1 \oplus E_2 \oplus \cdots \oplus E_k$ is closed.

Multiframes have the useful property that they lead to *multiplexed representations* of different signals. That is, if $(\mathcal{F}_1, \dots, \mathcal{F}_k)$ is a multiframe, then we will be able to reconstruct k different functions $g_1, \dots, g_k$ from the single sequence $\gamma_i =< g_1, f_k^1 > + \cdots + < g_k, f_i^k >$. In order to prove this, we introduce another way of describing multiframes.

With k frames $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_k$ we can construct the following set $\mathcal{F} = \{f_i, i \in \mathbf{I}\}$ "living" in the direct sum of k copies of H, $K = H \oplus H \oplus \cdots \oplus H$ and defined by $f_i = f_i^1 \oplus f_i^2 \oplus \cdots \oplus f_i^k$.

DEFINITION 2.11. We call $\mathcal{F}$ the *direct sum* of $\mathcal{F}_1, \dots, \mathcal{F}_k$ and we write $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_k$.

REMARK 2.12. Note that all these definitions (orthogonality, multiframe, direct sum) can be carried out to frames living in different Hilbert spaces, but having the same index set.

THEOREM 2.13. *Let $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k$ be k frames in H and let $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_k$ be their direct sum living in $K = H \oplus \cdots \oplus H$. Let us denote by $\pi_s : K \to H$ the canonical projection onto the sth component, $1 \leq s \leq k$. Thus $\mathcal{F}_s = \pi_s \mathcal{F}$.*

a) $(\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_k)$ is a multiframe in H iff $\mathcal{F}$ is a frame in K;

b) Let $\mathcal{G} = \{g_i, i \in \mathbf{I}\}$ be a frame in K. Then $\mathcal{G}_s = \pi_s \mathcal{G} = \{g_i^s = \pi_s g_i, i \in \mathbf{I}\}$ is a frame in H, for every $1 \leq s \leq k$; thus $\mathcal{G} = \mathcal{G}_1 \oplus \ldots \oplus \mathcal{G}_k$ is a direct of k-frames;

c) Suppose $\mathcal{F}$ is frame in K and let $\mathcal{F}'$ be a dual of $\mathcal{F}$. Let $\mathcal{F}_s^{'} = \pi_s \mathcal{F}'$, $1 \leq s \leq k$, i.e. $\mathcal{F}' = \mathcal{F}_1^{'} \oplus \ldots \oplus \mathcal{F}_k^{'}$. Then $\mathcal{F}_s^{'}$ is a dual of $\mathcal{F}_s$ and $\mathcal{F}_s^{'} \perp \mathcal{F}_l$, for $1 \leq s \neq l \leq k$. Conversely, suppose $(\mathcal{F}_1, \ldots, \mathcal{F}_k)$ and $(\mathcal{F}_1^{'}, \ldots, \mathcal{F}_k^{'})$ are multiframes in H such that $\mathcal{F}_s^{'}$ is dual to $\mathcal{F}_s$ and $\mathcal{F}_s^{'} \perp \mathcal{F}_l$, $\forall s \neq l$. Then $\mathcal{F}' = \mathcal{F}_1^{'} \oplus \cdots \mathcal{F}_k^{'}$ is a dual frame of $\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_k$.

d) Assume $\mathcal{F} = \mathcal{F}_1 \oplus \cdots \oplus \mathcal{F}_k$ is a frame in K, denote by $\tilde{\mathcal{F}} = \tilde{\mathcal{F}}_1 \oplus \cdots \tilde{\mathcal{F}}_k$ the standard dual of $\mathcal{F}$ in K, and let $\mathcal{F}^{'} = \mathcal{F}_1^{'} \oplus \cdots \oplus \mathcal{F}_k^{'}$ and $\mathcal{F}^d = \mathcal{F}_1^d \oplus \cdots \oplus \mathcal{F}_k^d$ be two dual frames of $\mathcal{F}$ frames of $\mathcal{F}$. Denote $\mathcal{H} = \mathcal{F}^{'} - \mathcal{F}^d$, $\mathcal{H} = \mathcal{H}_1 \oplus \cdots \oplus \mathcal{H}_k$, where $\mathcal{H}_l := \{f_i^{',l} - f_i^{d,l} \ ; \ i \in \mathbf{I}\}$. Then $\mathcal{H}_l$ is orthogonal to $\mathcal{F}_j$ in $l^2(\mathbf{I})$ sense, and $\mathcal{H}_l$ is orthogonal to $\tilde{\mathcal{F}}_j$ in $l^2(\mathbf{I})$ sense, for every $1 \leq j, l \leq k$.

PROOF. Firstly we prove b), then a), then c) and finally d).

b) Suppose $\mathcal{G}$ is a frame in K. Then there are positive constants A, B such that:

$$A\|f\|_K^2 \leq \sum_{i \in \mathbf{I}} | < f, g_i >_K |^2 \leq B\|f\|_K^2 \tag{2.7}$$

for every $f = f^1 \oplus \cdots \oplus f^k \in K$. Then, for the particular choice $f^l = \delta_{ls} f^s$ (δ_{ls} is the Kronecker symbol), (2.7) turns into: $A\|f^s\|^2 \leq \sum_{i \in \mathbf{I}} | < f^s, g_i^s > |^2 \leq B\|f^s\|^2$ which prove that $\mathcal{G}_s$ is a frame for H.

a) $\Leftarrow$ Suppose $\mathcal{F}$ is a frame in K. Point b) motivates that each $\mathcal{F}_s$ is a frame. We are going to prove that $E_l \cap E_s \{0\}$. Suppose there exists a $0 \neq c \in E_l \cap E_s$. Thus there are $z', z'' \in H$ such that $< z', f_i^l > = < z'', f_i^s > = c_i$, $\forall i \in \mathbf{I}$. Consider now the vector $z = z_1 \oplus \cdots \oplus z_k$ in K defined by $z_p = z'\delta_{pl} - z''\delta_{ps}$. Then $< z, f_i >_K = \sum_{p=1}^k < z_p, f_i^p >_H = 0$. Thus (2.7) turns into: $A(\|z'\|^2 + \|z''\|^2) \leq 0 \leq B(\|z'\|^2 + \|z''\|^2)$ which is impossible as long as $z', z'' \neq 0$. The contradiction comes from the assumption $E_s \cap E_l \neq \{0\}$. This proves that $E = E_1 \oplus \ldots \oplus E_k$ is a k-direct sum.

On the other hand, E is precisely the coefficient range associated to the frame $\mathcal{F}$ in K, thereore it is closed. Thus $(\mathcal{F}_1, \ldots, \mathcal{F}_k)$ is a multiframe in H.

$\Rightarrow$ Assume now that $(\mathcal{F}_1, \ldots, \mathcal{F}_k)$ is a multiframe in H. We are going to prove that $\mathcal{F}$ is a frame in K.

Let $E = E_1 \oplus E_2 \oplus \cdots \oplus E_k \subset l^2(\mathbf{I})$ and $X = E_1 \times E_2 \times \ldots \times E_k \subset (l^2(\mathbf{I}))^k$ endowed with the topology associated to the norm $\|(c_1, c_2, \ldots, c_k)\|_X = \|c_1\|_{E_1} + \|c_2\|_{E_2} + \cdots + \|c_k\|_{E_k}$. Let $q : E \to X$ denote the canonical decomposition $q(c) = (c_1, \ldots, c_k)$ such that $c = c_1 + \cdots + c_k$ and let $Q_s : E \to E_s$ be the projection onto the sth component, $q_s(c) = c_s$.

Firstly notice that q is bounded by the closed graph theorem. Therefore there exists a constant $C > 0$ such that:

$$\|c_1\|_{E_1} + \|c_2\|_{E_2} + \cdots + \|c_k\|_{E_k} \leq C\|c_1 + c_2 + \cdots + c_k\|_E$$

for every $c_1 \in E_1, c_2 \in E_2, \ldots, c_k \in E_k$. This proves, in particular, that each projection q_s is continuous.

Assume A_l, B_l are the frame bounds of the frame $\mathcal{F}_l$. Let $g = g^1 \oplus g^2 \oplus \cdots \oplus g^k \in K$ be an arbitrary k-vector. Let us denote by $c^s = \{< g^s, f_i^s > \}_{i \in \mathbf{I}} \in E_s \subset l^2(\mathbf{I})$ the coefficient sequence associated to g^s. In order to prove that $\mathcal{F}$ is a frame in K, we need to show that, for some constants A, B independent of g:

$$A(\|g^1\|^2 + \cdots \|g^k\|^2) \leq \|c^1 + \cdots + c^k\|^2 \leq k(\|c^1\|^2 + \cdots + \|c^k\|^2)$$

$$\leq kB_1\|g^1\|^2 + \cdots + kB_k\|g^k\|^2 \leq B(\|g^1\|^2 + \cdots + \|g^k\|^2)$$

for $B = k\max(B_1, \ldots, B_k)$. For the lower bound we use the continuity of q and the lower bounds of each $\mathcal{F}_s$:

$$\|c^1 + \cdots + c^k\|^2 \geq \frac{1}{C^2}(\|c^1\| + \cdots + \|c^k\|)^2 \geq \frac{1}{C^2}(A_1\|g^1\|^2 + \cdots + A_k\|g^k\|^2)$$

$$\geq A(\|g^1\|^2 + \cdots + \|g^k\|^2)$$

for $A = \frac{1}{C^2}\min(A_1, \ldots, A_k)$. Thus $\mathcal{F}$ is a frame in K.

c) Let $\mathcal{F} = \{f_i = f_i^1 \oplus \cdots \oplus f_i^k, i \in \mathbf{I}\}$ be the frame and $\mathcal{F}' = \{f_i^{'} = f_i^{'1} \oplus \cdots \oplus f_i^{'k}, i \in \mathbf{I}\}$ be the dual. The dual condition reads as:

$$g^1 \oplus \cdots \oplus g^k = \sum_{i \in \mathbf{I}} (\sum_{l=1}^{k} < g^l, f_i^l >) f_i^{'1} \oplus \cdots \oplus f_i^{'k} \tag{2.8}$$

for every $g^1, \ldots, g^k \in H$. Now, by choosing $g^l = \delta_{ls} g^s$ for a fixed s we get:

$$g^s = \sum_{i \in \mathbf{I}} < g^s, f_i^s > f_i^{'s} \quad 0 = \sum_{i \in \mathbf{I}} < g^s, f_i^s > f_i^{'l} \text{ for } l \neq s$$

This proves that $(T_l^{'})^* T_s = \delta_{ls} \cdot 1$ which means $\mathcal{F}_s^{'}$ is a dual of $\mathcal{F}_s$ and $\mathcal{F}_s^{'} \perp \mathcal{F}_l$, for $l \neq s$. For the converse $(T_l^{'})^* T_s = \delta_{ls} 1$ implies immediately (2.8) and thus the result is straightforward.

d) Applying c) it follows that both $\mathcal{F}_l^{'}$ and $\mathcal{F}_l^d$ are orthogonal to $\mathcal{F}_j$ (and hence $\tilde{\mathcal{F}}_j$) in $l^2(\mathbf{I})$ sense, for $j \neq l$. Hence $\mathcal{H}_l = \mathcal{F}_l^{'} - \mathcal{F}_l^d$ is orthogonal to both $\mathcal{F}_j$ and $\tilde{\mathcal{F}}_j$, $j \neq l$, as well. On the other hand, $\mathcal{F}_l^{'}$, $\mathcal{F}_l^d$ are both duals of $\mathcal{F}_l$. Hence:

$$g = \sum_{i \in \mathbf{I}} < g, f_i^{',l} > f_i = \sum_{i \in \mathbf{I}} < g, f_i^{d,l} > f_i^l \ , \ \forall g \in H$$

By taking the difference we get $\sum_{i \in \mathbf{I}} < g, f_i^{',l} - f_i^{d,l} > f_i^l = 0$, and thus $\mathcal{H}_l$ is orthogonal to $\mathcal{F}_l$, and hence $\tilde{\mathcal{F}}_l$ as well, in $l^2(\mathbf{I})$ sense. □

This result motivates the following definition:

Definition 2.14. Suppose $(\mathcal{F}_1, \ldots, \mathcal{F}_k)$ is a multiframe in H. A multiframe $(\mathcal{F}_1^{'}, \ldots, \mathcal{F}_k^{'})$ in H is called *dual (in the sense of multiframes)* to $(\mathcal{F}_1, \ldots, \mathcal{F}_k)$ if the frame $\mathcal{F}' = \mathcal{F}_1^{'} \oplus \ldots \oplus \mathcal{F}_k^{'}$ in K is dual to $\mathcal{F} = \mathcal{F}_1 \oplus \ldots \oplus \mathcal{F}_k$.

In this perspective, the point c) in Theorem 2.13 represents a biorthogonality condition for multiframes. With the help of the direct sum frame, we can introduce the corresponding equivalent of standard dual and associated tight frame:

DEFINITION 2.15. Suppose $(\mathcal{F}_1,\ldots,\mathcal{F}_k)$ is a multiframe in H. The *standard dual multiframe* of $(\mathcal{F}_1,\ldots,\mathcal{F}_k)$ is a multiframe denoted by $(\tilde{\mathcal{F}}_1,\ldots,\tilde{\mathcal{F}}_k)$ such that $\tilde{\mathcal{F}}=\tilde{\mathcal{F}}\oplus\cdots\oplus\tilde{\mathcal{F}}_k$ is the standard dual of $\mathcal{F}=\mathcal{F}_1\oplus\cdots\mathcal{F}_k$ in K. The *associated tight multiframe* of $(\mathcal{F}_1,\ldots,\mathcal{F}_k)$ is a multiframe denoted by $(\mathcal{F}_1^{\#},\ldots,\mathcal{F}_k^{\#})$ such that the direct sum $\mathcal{F}^{\#}=\mathcal{F}_1^{\#}\oplus\cdots\mathcal{F}_k^{\#}$ is the associated tight frame of $\mathcal{F}=\mathcal{F}_1\oplus\cdots\mathcal{F}_k$ in K.

REMARK 2.16. We stress out that the component $\tilde{\mathcal{F}}_l$, $1\le l\le k$ of a standard dual multiframe is not in general the standard dual frame of $\mathcal{F}_l$. However, the standard dual multiframe is the only dual multiframe for which the coefficient range of the direct sum $\tilde{\mathcal{F}}$ is the same with the coefficient range of $\mathcal{F}$. We notice also that the coefficient range of $\mathcal{F}^{\#}$ is the same with the coefficient range of $\mathcal{F}$.

2.3. Let us now consider $\mathcal{R}=\{f_i, i\in\mathbf{I}\}\subset H$ a Riesz basis for its span. A Riesz basis for its span $\mathcal{R}'=\{f_i^{'}, i\in\mathbf{I}\}$ is called a *pseudodual* of $\mathcal{R}$ if its analysis operator T' is a left inverse of the synthesis operator T^* associated to $\mathcal{R}$, i.e. $T'T^*=1_{l^2(\mathbf{I})}$. Explicitely, this means: $<f_i,f_j^{'}>=\delta_{ij}$, $\forall i,j\in\mathbf{I}$.

Note that, in general the pseudodual is not unique. However, when the span of $\mathcal{R}$ is the entire Hilbert space H, the pseudodual is unique and coincides with the standard dual associated to $\mathcal{R}$. A distingueshed pseudodual of $\mathcal{R}$ is the *standard dual* $\tilde{\mathcal{R}}=\{\tilde{f}_i, i\in\mathbf{I}\}$ of $\mathcal{R}$ when it is seen as a frame for its span R, i.e. $\tilde{f}_i=(S|_R)^{-1}f_i$ where $S|_R$ is the frame operator associated to $\mathcal{R}$ restricted to the span R.

One can also show that, if we denote by P_R the orthogonal projection onto the span $R\subset H$ then for any pseudodual $\mathcal{R}'=\{f_i^{'}, i\in\mathbf{I}\}$ of $\mathcal{R}$ we have $P_Rf_i^{'}=\tilde{f}_i$. The geometric interpretation of the standard dual of a Riesz basis for its span is the following: it is the only pseudodual included into the span.

An alternative formula of the standard dual is the following: consider the grammian operator $G:l^2(\mathbf{I})\to l^2(\mathbf{I})$ defined by $G(c)=\{<\sum_{i\in\mathbf{I}}c_if_i,f_j>\}_{j\in I}$. In the canonical basis of $l^2(\mathbf{I})$ given by $\{\delta_i\}_{i\in I}$ and introduced before, the matrix elements of G are $g_{ij}=<\delta_i,G\delta_j>=<f_i,f_j>$. Consider now the inverse of G, $G^{-1}:l^2(\mathbf{I})\to l^2(\mathbf{I})$ and its matrix elements $G_{ij}^{+}=<\delta_i,G^{-1}\delta_j>$. We claim that the standard dual is given by $\tilde{f}_i=\sum_{j\in I}g_{ij}^{+}f_j$. To see this notice that $\sum_{j\in I}g_{ij}^{+}f_j\in R$ and next $<\sum_{j\in I}g_{ij}^{+}f_j,f_l>=\sum_{j\in I}g_{ij}^{+}g_{jl}=\delta_{il}$.

For a frame, a dual provides a reconstruction formula for any vector in the Hilbert space, according to (2.3). For a Riesz basis for its span, a pseudodual gives a reconstruction formula for a sequence of coefficients, i.e.:

$$c=\{<\sum_{i\in\mathbf{I}}c_if_i,\tilde{f}_j>_H\}_{j\in\mathbf{I}}=\{<\sum_{i\in\mathbf{I}}c_i\tilde{f}_i,f_j>_H\}_{i\in\mathbf{I}}\ ,\ \forall c\in l^2(\mathbf{I}) \tag{2.9}$$

The second object linked to a Riesz basis for its span $\mathcal{R}$ is the *associated orthonormal set* $\mathcal{R}^{+}=\{f_i^{\#}\ ,\ i\in I\}$ defined by $f_i^{\#}=(S|_R)^{-1/2}f_i$. It corresponds to the associated tight frame described before. One can simply check that $\mathcal{R}^{\#}$ is an orthonormal basis for the span R. Alternatively, $\mathcal{R}^{\#}$ can be defined with the help of the grammian operator in the following way: Consider the inverse of the square root operator $G^{-1/2}:l^2(\mathbf{I})\to l^2(\mathbf{I})$ and its matrix elements $g_{ij}^{\#}=<\delta_i,G^{-1/2}\delta_j>$. Then one can show using the polar decomposition of T that $f_i^{\#}=\sum_{j\in I}g_{ij}^{\#}f_j$.

Consider now a finite number of Riesz basis for their span: $\mathcal{R}_1=\{f_i^1, i\in\mathbf{I}\}$,..., $\mathcal{R}_k=\{f_i^k, i\in\mathbf{I}\}$.

DEFINITION 2.17. The k-tuple $(\mathcal{R}_1, \ldots, \mathcal{R}_k)$ is called a *multi Riesz basis* if the set $\mathcal{R} = \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_k$ is a Riesz basis for its span.

We notice that the definition can be easily extended for Riesz bases for their span having different index sets.

DEFINITION 2.18. Two Riesz basis for their span $\mathcal{R}_1$ and $\mathcal{R}_2$ are called *orthogonal* if their spans are orhogonal, i.e. $R_1 \perp R_2$.

The following result summarizes the geometric conditions for a multi Riesz basis (the proof is an application of the closed graph theorem and is omitted):

THEOREM 2.19. *Suppose $\mathcal{R}_1 = \{f_i^1, i \in \mathbf{I}\}, \ldots, \mathcal{R}_k = \{f_i^k, i \in \mathbf{I}\}$ are k Riesz bases for their spans $R_1, \ldots, R_k$.*

a) The following two conditions are equivalent:

i) $(\mathcal{R}_1, \ldots, \mathcal{R}_k)$ is a multi Riesz basis;

ii) $R_l \cap R_s = \{0\}$, $\forall 1 \leq l \neq s \leq k$ and $R = R_1 \oplus \cdots \oplus R_k$ is closed;

Moreover, when $(\mathcal{R}_1, \ldots, \mathcal{R}_k)$ is a multi Riesz basis, $R = R_1 \oplus \cdots \oplus R_k$ is the span of $\cup_{l=1}^k \mathcal{R}_l$.

b) If $\mathcal{R}'$ is a pseudodual of $\mathcal{R} = \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_k$ then there exists a unique disjoint partition $\mathcal{R}' = \mathcal{R}_1' \cup \cdots \cup \mathcal{R}_k'$ such that $\mathcal{R}_l'$ is a pseudodual of $\mathcal{R}_l$ and $\mathcal{R}_l'$ is orthogonal to $\mathcal{R}_s$, for all $1 \leq l \neq s \leq k$.

REMARK 2.20. For $\mathcal{R}_1$ and $\mathcal{R}_2$, two Riesz bases for their span, the geoemtric condition ii) is equivalent to the angular condition $\alpha > 0$ where α is the minimum angle between their spans R_1, R_2, i.e.

$$\alpha = \inf_{\substack{0 \neq x_1 \in R_1 \\ 0 \neq x_2 \in R_2}} arccos \frac{|< x_1, x_2 >|}{\|x_1\| \cdot \|x_2\|}$$

The point b) of this Theorem motivates the following definition:

DEFINITION 2.21. A multi Riesz basis $(\mathcal{R}_1', \ldots, \mathcal{R}_k')$ is called *dual* of a given multi Riesz basis $(\mathcal{R}_1, \ldots, \mathcal{R}_k)$ if for every $1 \leq s \leq k$, $\mathcal{R}_s'$ is a pseudodual of $\mathcal{R}_s$ and for every $1 \leq s \neq l \leq k$, $\mathcal{R}_s'$ is orthogonal to $\mathcal{R}_l$.

Similar to the one Riesz basis case we introduce the following definition:

DEFINITION 2.22. Suppose $(\mathcal{R}_1, \ldots, \mathcal{R}_k)$ is a multi Riesz basis for H. The *standard dual* of $(\mathcal{R}_1, \ldots, \mathcal{R}_k)$ is a multi Riesz basis $(\tilde{\mathcal{R}}_1, \ldots, \tilde{\mathcal{R}}_k)$ such that $\cup_{l=1}^k \tilde{\mathcal{R}}_l$ is the standard dual of $\cup_{l=1}^k \mathcal{R}_l$. The *associated orthonormal multi set* of $(\mathcal{R}_1, \ldots, \mathcal{R}_k)$ is a multi Riesz basis $(\mathcal{R}_1^{\#}, \ldots, \mathcal{R}_k^{\#})$ such that $\cup_{l=1}^k \mathcal{R}_l^{\#}$ is the associated orthonormal set of $\cup_{l=1}^k \mathcal{R}_l$.

REMARK 2.23. We point out the fact that the union sets $\cup_{l=1}^k \mathcal{R}_l$, $\cup_{l=1}^k \tilde{\mathcal{R}}_l$, $\cup_{l=1}^k \mathcal{R}_l^{\#}$ are considered ordered sets having the same index set, for instance $\{1, 2, \ldots, k\} \times \mathbf{Z} \times \mathbf{Z}$. This is an important remark because it eliminates any possible confusion (and nonuniqueness) that the definition may rise.

REMARK 2.24. An equivalent definition of the standard dual is to say that it is the only dual multi Riesz basis which is included in the span R of the initial multi Riesz basis.

REMARK 2.25. The associated orthonormal multi set is an orthonormal basis of the span R.

In the next section we shall concentrate on the multiframes and multi Riesz bases of the Weyl-Heisenberg type.

3. Weyl-Heisenberg Multiframe and Multi Riesz basis

For two real numbers $a, b \in \mathbf{R}$ we introduce a unitary operator:

$$U(a,b) : L^2(\mathbf{R}) \to L^2(\mathbf{R}) \ , \ U(a,b)f(x) = e^{-i\pi ab}e^{2\pi iax}f(x-b)$$

Let us consider two positive real munbers α, β and a function $g \in L^2(\mathbf{R})$. With this data we construct the following functions indexed by integers $m, n \in \mathbf{Z}$:

$$g_{mn;\alpha\beta}(x) := U(m\alpha, n\beta)g(x) = e^{2\pi i m\alpha x - i\pi mn\alpha\beta}g(x-n\beta) \tag{3.1}$$

and set

$$\mathcal{WH}_{g;\alpha\beta} = \{g_{mn;\alpha,\beta} \ ; \ (m,n) \in \mathbf{Z}^2\} \tag{3.2}$$

If the set $\mathcal{WH}_{g;\alpha\beta}$ is a frame it is called a *Weyl-Heisenberg frame*. Note the extra phase factor $-i\pi mn\alpha\beta$ that appears in (3.1), if one compares with the standard Weyl-Heisenberg frame theory ([**HaWa89**] or [**Daub90**]).

Likewise, if $\mathcal{WH}_{g;\alpha\beta}$ is a Riesz basis (for $L^2(\mathbf{R})$), or a Riesz basis for its span, it will be called *Weyl-Heisenberg Riesz basis* or *Weyl-Heisenberg Riesz basis for its span* and when it is a Bessel sequence it will be called a *Weyl-Heisenberg Bessel sequence.*

We shall denote by $T_{g;\alpha,\beta}$ the analysis operator associated to a Weyl - Heisenberg Bessel sequence, i.e. $T_{g;\alpha,\beta} : L^2(\mathbf{R}) \to l^2(\mathbf{Z}^2)$, $T_{g;\alpha,\beta}(f) = \{< f, g_{mn;\alpha\beta} >\}_{m,n\in\mathbf{Z}}$. Fot the convenience of the reader, the known results and of interest for us will be summarized in the following theorem (see [**Jans95**] and [**DLL95**]):

THEOREM 3.1. *Let $\alpha, \beta > 0$ and $g \in L^2(\mathbf{R})$.*

I. *Suppose $\mathcal{WH}_{g;\alpha\beta}$ is a frame. Let us denote by S the frame operator, i.e. $S = \sum_{m,n} < \cdot, g_{mn;\alpha\beta} > g_{mn;\alpha\beta}$. Then:*
   (a) $\alpha \cdot \beta \leq 1$
   (b) *$\mathcal{WH}_{g;\alpha\beta}$ is a Riesz basis (for $L^2(\mathbf{R})$) if and only if $\alpha\beta = 1$*
   (c) $U(m\alpha, n\beta)S = SU(m\alpha, n\beta)$
   (d) *There exists $\tilde{g} \in L^2(\mathbf{R})$ such that $\mathcal{WH}_{\tilde{g};\alpha,\beta}$ is the standard dual of $\mathcal{WH}_{g;\alpha\beta}$; furthermore $\tilde{g} = S^{-1}g$;*
   (e) *There exists $g^{\#} \in L^2(\mathbf{R})$ such that $\mathcal{WH}_{g^{\#};\alpha,\beta}$ is the associated tight frame of $\mathcal{WH}_{g;\alpha\beta}$; furthermore $g^{\#} = S^{-1/2}g$;*

II. *Suppose $\mathcal{WH}_{g;\alpha\beta}$ is a Riesz basis for its span. Then:*
   (a) $\alpha\beta \geq 1$
   (b) *There exists $\tilde{g} \in L^2(\mathbf{R})$ such that $\mathcal{WH}_{\tilde{g};\alpha,\beta}$ is the standard dual of $\mathcal{WH}_{g;\alpha\beta}$;*
   (c) *There exists $g^{\#} \in L^2(\mathbf{R})$ such that $\mathcal{WH}_{g^{\#};\alpha,\beta}$ is the associated orthonormal set of $\mathcal{WH}_{g;\alpha\beta}$;*
   (d) *$V_{mn} = T_{g^{\#};\alpha,\beta}U(m\alpha, n\beta)T^*_{g^{\#};\alpha,\beta}$ is a unitary operator on $l^2(\mathbf{Z}^2)$ and $GV_{mn} = V_{mn}G$ for every $m, n \in \mathbf{Z}$;*

III. (a) *If $T_{g;\alpha,\beta}$ is bounded, so is $T_{g;\frac{1}{\beta},\frac{1}{\alpha}}$;*

(b) *Let* $f, h \in L^2(\mathbf{R})$ *such that* $T_{g;\alpha,\beta}$, $T_{f;\alpha,\beta}$, $T_{h;\frac{1}{\beta},\frac{1}{\alpha}}$ *are all bounded. Then:*

$$T^*_{f;\alpha,\beta} T_{g;\alpha,\beta} h = \frac{1}{\alpha\beta} T^*_{h;\frac{1}{\beta},\frac{1}{\alpha}} T_{g;\frac{1}{\beta},\frac{1}{\alpha}} f \tag{3.3}$$

(the Wexler-Raz identity*).*

IV. $\mathcal{WH}_{g;\alpha\beta}$ *is a frame if and only if* $\mathcal{WH}_{g;\frac{1}{\beta},\frac{1}{\alpha}}$ *is a Riesz basis for its span;*

V. *Suppose* $\mathcal{WH}_{g;\alpha\beta}$ *is a frame and* $\mathcal{WH}_{g;\frac{1}{\beta},\frac{1}{\alpha}}$ *a Riesz basis for its span.*

(a) $\mathcal{WH}_{g';\alpha,\beta}$ *is a dual of* $\mathcal{WH}_{g;\alpha\beta}$ *if and only if* $\mathcal{WH}_{g';\frac{1}{\beta},\frac{1}{\alpha}}$ *is a pseudodual of* $\mathcal{WH}_{g;\frac{1}{\beta},\frac{1}{\alpha}}$*;*

(b) $\mathcal{WH}_{\tilde{g};\alpha,\beta}$ *is the standard dual of* $\mathcal{WH}_{g;\alpha\beta}$ *if and only if* $\mathcal{WH}_{\tilde{g};\frac{1}{\beta},\frac{1}{\alpha}}$ *is the standard dual of* $\mathcal{WH}_{g;\frac{1}{\beta},\frac{1}{\alpha}}$*;*

(c) $\mathcal{WH}_{g\#;\alpha,\beta}$ *is the associated tight frame of* $\mathcal{WH}_{g;\alpha\beta}$ *if and only if* $\mathcal{WH}_{g\#;\frac{1}{\beta},\frac{1}{\alpha}}$ *is the associated orthonormal set of* $\mathcal{WH}_{g;\frac{1}{\beta},\frac{1}{\alpha}}$*;*

Let $k \geq 1$ be an integer. We denote by $K = L^2(\mathbf{R}) \oplus \ldots \oplus L^2(\mathbf{R})$ the k-times direct sum of the basic Hilbert space endowed with the scalar product:

$$< f_1 \oplus \ldots \oplus f_k, h_1 \oplus \ldots \oplus h_k > = \sum_{j=1}^{k} < f_j, h_j > \tag{3.4}$$

The canonical projection onto the jth component of K is denoted by π_j; thus $\pi_j : K \to L^2(\mathbf{R})$ by $\pi_j(f_1 \oplus \ldots \oplus f_k) = f_j$.

For two vector parameters $\underline{a} = (a_1, \ldots, a_k)$, $\underline{b} = (b_1, \ldots, b_k)$ we introduce the following unitary operator:

$$U(\underline{a}, \underline{b}) : K \to K \ , \ U(\underline{a}, \underline{b}) = \bigoplus_{j=1}^{k} U(a_j, b_j)\pi_j$$

One can easily check that the adjoint is $U(\underline{a}, \underline{b})^* = U(-\underline{a}, -\underline{b})$.

Consider now a vector $\underline{g} = g^1 \oplus \ldots \oplus g^k \in K$ and two positive vector parameters $\underline{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \mathbf{R}^k_+$ and $\underline{\beta} = (\beta_1, \ldots, \beta_k) \in \mathbf{R}^k_+$. We construct now three objects using the previous unitary operator and the discrete lattices

$$\mathcal{L}^1_{\underline{\alpha},\underline{\beta}} = \{(m\underline{\alpha}, n\underline{\beta}); (m, n) \in \mathbf{Z}^2\} \subset \mathbf{R}^{2k}$$

and

$$\mathcal{L}^2_{\underline{\alpha},\underline{\beta}} = \{m\alpha_1 \underline{e}_1 + n\beta_1 \underline{e}_2 \ ; \ m, n \in \mathbf{Z}\} \cup \{m\alpha_2 \underline{e}_3 + n\beta_2 \underline{e}_4 \ ; \ m, n \in \mathbf{Z}\} \cup \ldots$$

$$\cup \{m\alpha_k \underline{e}_{2k-1} + n\beta_k \underline{e}_{2k} \ ; \ m, n \in \mathbf{Z}\} \subset \mathbf{R}^{2k}$$

(where $\underline{e}_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ is the unit vector having a 1 on the j'th position): (a) a *Weyl-Heisenberg set of type I*

$$\mathcal{WH}_{\underline{g};\underline{\alpha},\underline{\beta}} = \{U(m\underline{\alpha}, n\underline{\beta})\underline{g} \ ; \ (m, n) \in \mathbf{Z}^2\} \tag{3.5}$$

(b) a *Weyl-Heisenberg set of type II*

$$\mathcal{R}_{\underline{g};\underline{\alpha};\underline{\beta}} = \bigcup_{l=1}^{k} \mathcal{WH}_{g^l;\alpha_l,\beta_l} \tag{3.6}$$

and (c) a k-tuple called *Weyl-Heisenberg k-tuple*

$$(\underline{g}; \underline{\alpha}, \underline{\beta}) = (\mathcal{WH}_{g_1;\alpha_1,\beta_1}, \ldots, \mathcal{WH}_{g_k;\alpha_k,\beta_k}) \tag{3.7}$$

The Weyl-Heisenberg set of type I is a coherent set in the sense of [**Pere72**]. Since we will not deal in this paper with other coherent sets (associated to other Lie groups), we shall use the term *coherent* as meaning *of Weyl-Heisenberg type I or II* (or of form 3.5 or 3.6).

If the Weyl-Heisenberg k-tuple is a multiframe or a multi Riesz basis, we shall call it a Weyl-Heisenberg multi frame, respectively a *Weyl-Heisenberg multi Riesz basis*. Some special cases of Weyl-Heisenberg sets and k-tuples will be useful later:

- A Weyl-Heisenberg set or k-tuple is called *super-coherent* if:

$$\alpha_1\beta_1 = \alpha_2\beta_2 = \cdots = \alpha_k\beta_k \tag{3.8}$$

- A Weyl-Heisenberg set or k-tuple is called *ultra-coherent* if:

$$\alpha_1 = \alpha_2 = \cdots = \alpha_k \text{ and } \beta_1 = \beta_2 = \cdots = \beta_k \tag{3.9}$$

- A Weyl-Heisenberg set or k-tuple is called *rational* if:

$$M_1\alpha_1 = M_2\alpha_2 = \cdots = M_k\alpha_k \text{ and } N_1\beta_1 = N_2\beta_2 = \cdots = N_k\beta_k \tag{3.10}$$

  for some integers $M_1, M_2, \ldots, M_k, N_1, N_2, \ldots, N_k$.
- A Weyl-Heisenberg k-tuple is called *orthogonal* if either it is a multiframe and for every $1 \leq l \neq s \leq k$, $\mathcal{WH}_{g^l;\alpha_l,\beta_l}$ is orthogonal to the Weyl-Heisenberg frame $\mathcal{WH}_{g^s;\alpha_s,\beta_s}$, or it is a multi Riesz basis and for every $1 \leq l \neq s \leq k$, $\mathcal{WH}_{g^l;\alpha_l,\beta_l}$ is orthogonal to the Weyl-Heisenberg Riesz basis for its span $\mathcal{WH}_{g^s;\alpha_s,\beta_s}$.

We study now necessary and sufficient conditions to have orthogonal Weyl-Heisenberg frames and, respectively, orthogonal Weyl-Heisenberg Riesz bases for their span. Such conditions are useful later on when we shall look for coherent k-tuple (multiframe or multi Riesz basis) having coherent dual.

THEOREM 3.2. *Let g^1, g^2 be two functions in $L^2(\mathbf{R})$ and $\alpha_1, \alpha_2, \beta_1, \beta_2$ positive real numbers. Let us denote by $R_1 = \overline{span\, \mathcal{WH}_{g^1;\alpha_1,\beta_1}}$ and $R_2 = \overline{span\, \mathcal{WH}_{g^2;\alpha_2,\beta_2}}$ the closure of the coherent sets spans.*

Case 1. *$\frac{\alpha_1}{\alpha_2}$ and $\frac{\beta_1}{\beta_2}$ are both rational. Let $\frac{\alpha_1}{\alpha_2} = \frac{P_1}{P_2}$ and $\frac{\beta_1}{\beta_2} = \frac{Q_1}{Q_2}$ be the irreducible representations of these ratios with $P_1, P_2, Q_1, Q_2 \in \mathbf{N}$. Set $v_1 = \frac{P_1}{\alpha_1}$ and $v_2 = \frac{\beta_1}{Q_1}$. Then R_1 is orthogonal to R_2 (in $L^2(\mathbf{R})$ sense) if and only if:*

$$\sum_n \overline{g^1(t+nv_1)}g^2(t+nv_1+kv_2) = 0 \tag{3.11}$$

for almost every $t \in [0, v_1]$ and every $k \in \mathbf{Z}$.

Case 2. *$\frac{\alpha_1}{\alpha_2}$ is rational and $\frac{\beta_1}{\beta_2}$ is irrational. Let $\frac{\alpha_1}{\alpha_2} = \frac{P_1}{P_2}$ be the irreducible representation of $\frac{\alpha_1}{\alpha_2}$ with $P_1, P_2 \in \mathbf{N}$. Set $v = \frac{\alpha_1}{P_1}$. Then R_1 is orthogonal to R_2 (in $L^2(\mathbf{R})$ sense) if and only if*

$$\overline{\hat{g}^1(\xi)}\hat{g}^2(\xi + 2\pi m v) = 0 \tag{3.12}$$

for almost every $\xi \in \mathbf{R}$ and every $m \in \mathbf{Z}$.

Case 3. *$\frac{\alpha_1}{\alpha_2}$ is irrational and $\frac{\beta_1}{\beta_2}$ is rational. Let $\frac{\beta_1}{\beta_2} = \frac{Q_1}{Q_2}$ be the irreducible representation of $\frac{\beta_1}{\beta_2}$ with $Q_1, Q_2 \in \mathbf{N}$. Set $v = \frac{\beta_1}{Q_1}$. Then R_1 is orthogonal to R_2 (in $L^2(\mathbf{R})$ sense) if and only if*

$$\overline{g^1(x)}g^2(x+nv) = 0 \tag{3.13}$$

for almost every $x \in \mathbf{R}$ and every $n \in \mathbf{Z}$.

Case 4. *$\frac{\alpha_1}{\alpha_2}$ and $\frac{\beta_1}{\beta_2}$ are both irrational. Then R_1 is orthogonal to R_2 (in $L^2(\mathbf{R})$ sense) if and only if $\|g^1\|\cdot\|g^2\|=0$.*

In particular, if $\mathcal{WH}_{g^1;\alpha_1,\beta_1}$ and $\mathcal{WH}_{g^2;\alpha_2,\beta_2}$ are orthogonal Riesz bases for their span, then at least one of $\frac{\alpha_1}{\alpha_2}$ or $\frac{\beta_1}{\beta_2}$ is rational.

THEOREM 3.3. *Let g^1, g^2 be two functions in $L^2(\mathbf{R})$ and let $\alpha_1,\alpha_2,\beta_1,\beta_2$ be positive real numbers. Suppose $\mathcal{WH}_{g^1;\alpha_1,\beta_1}$ and $\mathcal{WH}_{g^2;\alpha_2,\beta_2}$ are Bessel sequences. Let us denote by E_1, respectively E_2, the closure of the coefficient ranges, i.e. $E_1=\overline{Ran\,T_1}$, $E_2=\overline{Ran\,T_2}$ where $T_1=T_{g^1;\alpha_1,\beta_1}$ and $T_2=T_{g^2;\alpha_2,\beta_2}$.*

Case 1. *$\alpha_1\beta_1-\alpha_2\beta_2$ is rational. Let $\frac{\alpha_1\beta_1-\alpha_2\beta_2}{2}=\frac{P}{Q}$ be the irreducible representation with $P\in\mathbf{Z}$ and $Q\in\mathbf{N}$, and, since P and Q are relatively prime let $m_0,n_0\in\mathbf{Z}$ be their cofactora: $n_0P+m_0Q=1$. Set $v_1=Q\beta_1$, $v_2=Q\beta_2$ and $v_3=\frac{m_0Q-n_0P}{Q\alpha_2}$. Then E_1 is orthogonal to E_2 (in $l^2(\mathbf{Z}^2)$ sense) if and only if:*

$$\sum_k \overline{g^1(x+kv_1)}g^2(\frac{\alpha_1}{\alpha_2}x+kv_2+lv_3)=0 \tag{3.14}$$

for almost every $x\in\mathbf{R}$ and every $l\in\mathbf{Z}$.

Case 2. *$\alpha_1\beta_1-\alpha_2\beta_2$ is irrational. Then E_1 is orthogonal to E_2 (in $l^2(\mathbf{Z}^2)$ sense) if and only if $\|g^1\|\cdot\|g^2\|=0$.*

In particular, if $\mathcal{WH}_{g^1;\alpha_1,\beta_1}$ is an orthogonal frame to $\mathcal{WH}_{g^2;\alpha_2,\beta_2}$ then necessary $\alpha_1\beta_1-\alpha_2\beta_2$ is rational.

The proofs of these two theorems are based on the following two lemmas:

LEMMA 3.4. *Consider a_1 and a_2 two real numbers such that $\frac{a_1}{a_2}$ is rational. Let $\frac{a_1}{a_2}=\frac{P}{Q}$ be the irreducible representation and n_0,m_0 the cofactors of P,Q, i.e. $n_0P+m_0Q=1$ with $n_0,m_0,P\in\mathbf{Z}$ and $Q\in\mathbf{N}$. Set $a=\frac{a_1}{P}$. Then:*

$$\{na_1+ma_2\ ;\ n,m\in\mathbf{Z}\}=\{la\ ;\ l\in\mathbf{Z}\}$$

and

$$\{(n,m)\ ;\ n,m\in\mathbf{Z}\text{ and }na_1+ma_2=la\}=\{(ln_0+kQ,lm_0-kP)\ ;\ k\in\mathbf{Z}\}$$

The second lemma has been used in various forms in frame computations (see proof of Lemma 3.2 in [**DLL95**] or Theorem 2 in [**ChSh93b**] or proof of Theorem 4.1.5 in [**HaWa89**]). The author is grateful to I.Daubechies for a discussion of this relation. It represents a weaker form of Poisson summation formula under some conditions:

LEMMA 3.5 (Weak-L^2 Poisson summation formula against *nice* functions). *Suppose $f_1,f_2\in L^2(\mathbf{R})$ and $g_1,g_2\in S$ (the class of Schwartz functions). Then:*

$$\sum_n\int\int dx_1dx_2 f_1(x_1)g_1(x_1)\overline{f_2(x_2)}\,\overline{g_2(x_2)}e^{2\pi in(x_2-x_1)}$$

$$=\sum_n\int dx f_1(x)g_1(x)\overline{f_2(x+n)}\,\overline{g_2(x+n)}$$

$$=\int dx f_1(x)g_1(x)\sum_n\overline{f_2(x+n)}\,\overline{g_2(x+n)}$$

where the integrals and the last two sums converge absolutely.

The proof uses Parseval's relation and, for the absolute convergence, the Cauchy-Schwartz inequality. We point out that the presence of both g_1 and g_2 is crucial for the absolute convergence. However one can relax the smoothness conditions on g_1, g_2 by asking, for instance, to be piecewise continuous and compactly supported functions.

PROOF OF THEOREM 3.2. R_1 orthogonal to R_2 is equivalent with:

$$< g^1_{m_1n_1;\alpha_1\beta_1}, g^2_{m_2n_2;\alpha_2\beta_2} >= 0 \ , \ \forall m_1, m_2, n_1, n_2 \in \mathbf{Z}$$

Explicitely:

$$\int_{-\infty}^{\infty} e^{2\pi i(m_1\alpha_1 - m_2\alpha_2)x} g^1(x)\overline{g^2(x + n_1\beta_1 - n_2\beta_2)}dx = 0$$

or, using Fourier transform and Plancherel relation:

$$\int_{-\infty}^{\infty} e^{-i\xi(n_1\beta_1 - n_2\beta_2)} \hat{g}^1(\xi)\overline{\hat{g}^2(\xi + 2\pi(m_1\alpha_1 - m_2\alpha_2))}d\xi = 0$$

Let us denote by $F_1(x) = g^1(x)\overline{g^2(x + n_1\beta_1 - n_2\beta_2)}$ and $F_2(\xi) = \hat{g}^1(\xi)\overline{\hat{g}^2(\xi + 2\pi(m_1\alpha_1 - m_2\alpha_2))}$. The orthogonality condition is then equivalent to:

$$\hat{F}_1(2\pi(m_2\alpha_2 - m_1\alpha_1)) = 0 \text{ respectively } \hat{F}_2(n_1\beta_1 - n_2\beta_2) = 0$$

On the other hand, since $F_1, F_2 \in L^1(\mathbf{R})$, $\hat{F}_1, \hat{F}_2$ are continuous.

Suppose $\frac{\alpha_1}{\alpha_2}$ is irrational. Then $\{2\pi(m_2\alpha_2 - m_1\alpha_1); m_1, m_2 \in \mathbf{Z}\}$ is dense in $\mathbf{R}$ and then $\hat{F}_1 \equiv 0 \Rightarrow F_1(x) = 0$ a.e. $x \in \mathbf{R}$. If furthermore $\frac{\beta_1}{\beta_2}$ is irrational the set $\{n_1\beta_1 - n_2\beta_2; n_1, n_2 \in \mathbf{Z}\}$ is also dense in $\mathbf{R}$ and the condition $g^1(x)\overline{g^2(x + n_1\beta_1 - n_2\beta_2)} = 0, \forall n_1, n_2 \Rightarrow g^1(x) = 0$ a.e. x or $g^2(x) = 0$ a.e. x, which is equivalent with $\|g^1\| \cdot \|g^2\| = 0$ and hence Case 4 is proved. If $\frac{\beta_1}{\beta_2}$ is rational (with $\frac{\alpha_1}{\alpha_2}$ irrational), bu using Lemma 3.4, $\{n_1\beta_1 - n_2\beta_2; n_1, n_2 \in \mathbf{Z}\} = \{nv, n \in \mathbf{Z}\}$ with $v = \frac{\beta_1}{Q_1}$ as in hypothesis. Thus we obtain (3.13) and Case 3 is done.

Suppose now $\frac{\beta_1}{\beta_2}$ is irrational. Using a similar argument as before, but this time for F_2, we obtain (3.12).

Finally, consider Case 1 when both ratios $\frac{\alpha_1}{\alpha_2}$ and $\frac{\beta_1}{\beta_2}$ are rational. Then the orthogonality condition turns into:

$$\int_{-\infty}^{\infty} e^{2\pi i \frac{m}{v_1}x} g^1(x)\overline{g^2(x - nv_2)}dx = 0 \ , \ \forall m, n \in \mathbf{Z}$$

By periodizing (note that $g^1, g^2 \in L^2(\mathbf{R})$ and therefore we can commute the sum with the integral) we get:

$$\int_0^{v_1} e^{2\pi i \frac{m}{v_1}x} \sum_k g^1(x + kv_1)\overline{g^2(x + kv_1 - nv_2)}dx = 0 \ , \ \forall m, n \in \mathbf{Z}$$

Since $\{e^{2\pi i \frac{m}{v_1}x} \ , \ m \in \mathbf{Z}\}$ is complete in $L^2[0, v_1]$ we obtain that the integrand should be zero a.e. and thus (3.11) is established. □

PROOF OF THEOREM 3.3. The orthogonality condition is equivalent with:

$$\sum_{m,n} < F, g^1_{mn;\alpha_1\beta_1} >< g^2_{mn;\alpha_2\beta_2}, H >= 0 \ , \ \forall F, H \in L^2(\mathbf{R})$$

a) Suppose $\gamma = \alpha_1\beta_1 - \alpha_2\beta_2$ is irrational.

Consider $F = U(m_0\alpha_1, n_0\beta_1)f$ and $H = U(m_0\alpha_2, n_0\beta_2)h$. Using $U(a,b)^* = U(-a,-b)$ and $U(a_1,b_1)U(a_2,b_2) = e^{i\pi(a_1b_2-a_2b_1)}U(a_1+a_2,b_1+b_2)$ we get:

$$\sum_{m,n} e^{i\pi(m_0n-n_0m)(\alpha_1\beta_1-\alpha_2\beta_2)} < f, g^1_{mn;\alpha_1\beta_1} >< g^2_{mn;\alpha_2\beta_2}, h >= 0$$

for every $f,h \in L^2(\mathbf{R})$ and $m_0,n_0 \in \mathbf{Z}$. Let us denote by $c_m = \sum_n e^{i\pi m_0 n\gamma} < f, g^1_{mn;\alpha_1\beta_1} >< g^2_{mn;\alpha_2\beta_2}, h >$. We have $\{c_m\}_{m\in\mathbf{Z}} \in l^1(\mathbf{Z})$ since $\mathcal{WH}_{g^1;\alpha_1,\beta_1}$ and $\mathcal{WH}_{g^2;\alpha_2,\beta_2}$ are Bessel sequence. Thus $\sum_m c_m e^{2\pi i m\frac{n_0\gamma}{2}} = 0$ for every $n_0 \in \mathbf{Z}$. The function $t \mapsto \sum_m c_m e^{2\pi i m t}$ is continuous on $[0,1]$ and since $\{\frac{n_0\gamma}{2} mod 1, n_0 \in \mathbf{Z}\}$ is dense in $[0,1]$ (the ergodic property of the irrational linear flow on T^2) we obtain that $\sum_m c_m e^{2\pi i m t} \equiv 0$, which means $c_m = 0$, $\forall m$. Applying the same trick to $\sum_n d_n^{(m)} e^{2\pi i n\frac{m_0\gamma}{2}} = 0$ with $d_n^{(m)} =< f, g^1_{mn;\alpha_1,\beta_1} >< g^2_{mn;\alpha_2,\beta_2}, h >$ we get $d_n^{(m)} = 0$ for every $m,n \in \mathbf{Z}$ and $f,h \in L^2(\mathbf{R})$. Thus for $f = g^1$, $h = g^2$ and $m = n = 0$ we have $\|g^1\| \cdot \|g^2\| = 0$.

b) Suppose now $\gamma = \alpha_1\beta_1 - \alpha_2\beta_2$ is rational and P,Q as in hypothesis. Consider $f,h \in S$ (the class of rapidly decreasing $\mathcal{C}^\infty$ functions). Then:

$$E_{f,h} = \sum_{m,n} < f, g^1_{mn;\alpha_1\beta_1} >< g^2_{mn;\alpha_2\beta_2}, h >$$

$$= \sum_n \sum_m \int\int dx_1 dx_2 f(x_1)\overline{h(x_2)}\,\overline{g^1(x_1-n\beta_1)}g^2(x_2-n\beta_2)e^{2\pi i m(\alpha_2x_2-\alpha_1x_1+\frac{n}{2}\gamma)}$$

Using Lemma 3.5 we get:

$$E_{f,h} = \frac{1}{\alpha_2}\sum_{m,n}\int dx f(x)\overline{h(\frac{\alpha_1}{\alpha_2}x - \frac{n\gamma}{2\alpha_2} + \frac{m}{\alpha_2})}\,\overline{g^1(x-n\beta_1)}g^2(\frac{\alpha_1}{\alpha_2}x - \frac{n\gamma}{2\alpha_2} + \frac{m}{\alpha_2} - n\beta_2)$$

Now, the orthogonality condition is equivalent to $E_{f,h} = 0\ \forall f,h \in S$ which implies (with $h_d(x) = \overline{h(\frac{\alpha_1}{\alpha_2}x)}$):

$$\sum_{m,n} h_d(x + \frac{n\gamma}{2\alpha_1} + \frac{m}{\alpha_1})\overline{g^1(x+n\beta_1)}g^2(\frac{\alpha_1}{\alpha_2}x + \frac{n(\alpha_1\beta_1+\alpha_2\beta_2)}{2\alpha_2} + \frac{m}{\alpha_2}) = 0 \tag{3.15}$$

for every $h_d \in S$ and almost every $x \in \mathbf{R}$.

The next step is to change the summation according to the argument of h_d. Using Lemma 3.4 we have:

$$\{\frac{n\gamma}{2\alpha_1} + \frac{m}{\alpha_1}\ ;\ m,n \in \mathbf{Z}\} = \{l\frac{1}{Q\alpha_1}\ ;\ l \in \mathbf{Z}\}$$

and

$$I_l = \{(m,n)\ ;\ \frac{n\gamma}{2\alpha_1} + \frac{m}{\alpha_1} = \frac{l}{Q\alpha_1}\} = \{(lm_0 - kP, ln_0 + kQ)\ ;\ k \in \mathbf{Z}\}$$

with P,Q,m_0,n_0 introduced in the hypothesis. Then (3.15) turns into:

$$\sum_l h_d(x + \frac{l}{Q\alpha_1})\sum_k \overline{g^1(x+(ln_0+kQ)\beta_1)}g^2(\frac{\alpha_1}{\alpha_2}x + \frac{(ln_0+kQ)(\alpha_1\beta_1+\alpha_2\beta_2)}{2\alpha_2}$$

$$+\frac{lm_0-kP}{\alpha_2}) = 0$$

Since h_d is arbitrary in S we obtain after the change of variable $x + ln_0 = y$ the equivalent orthogonality condition as:

$$\sum_k \overline{g^1(y + kQ\beta_1)} g^2(\frac{\alpha_1}{\alpha_2} y + kQ\beta_2 + \frac{l}{Q\alpha_2}(m_0Q - n_0P)) = 0$$

for almost every $y \in \mathbf{R}$ and every $l \in \mathbf{Z}$ which ends the proof. $\square$

In the same context it is worth mentioning the following result:

THEOREM 3.6. *Let α, β be positive real numbers and $g^1, g^2 \in L^2(\mathbf{R})$ such that $\mathcal{WH}_{g^1;\alpha,\beta}$ and $\mathcal{WH}_{g^2;\alpha,\beta}$ are frames (or, equivalently, $\mathcal{WH}_{g^1;\frac{1}{\beta},\frac{1}{\alpha}}$ and $\mathcal{WH}_{g^2;\frac{1}{\beta},\frac{1}{\alpha}}$ are Riesz bases for their span). Then, the following two statements are equivalent:*

1. *The frame $\mathcal{WH}_{g^1;\alpha,\beta}$ is orthogonal to the frame $\mathcal{WH}_{g^2;\alpha,\beta}$;*
2. *The Riesz basis for its span $\mathcal{WH}_{g^1;\frac{1}{\beta},\frac{1}{\alpha}}$ is orthogonal to the Riesz basis for its span $\mathcal{WH}_{g^2;\frac{1}{\beta},\frac{1}{\alpha}}$;*

PROOF. The result follows easily from the Wexler-Raz identity (3.3) observing that the orthogonality of two frames is equivalent to $T^*_{g^1;\alpha,\beta}T_{g^2;\alpha,\beta} = 0$ whereas the orthogonality of two Riesz bases for their span is equivalent with $T_{g^2;\frac{1}{\beta},\frac{1}{\alpha}}g^1 = 0$. $\square$

After these preliminary results, we are able to atack the main problem of this section, namely to give sufficient conditions for a coherent k-tuple to have a coherent (standard) dual.

THEOREM 3.7. *Let $k \geq 1$ be an integer and $K = L^2(\mathbf{R}) \oplus \cdots \oplus L^2(\mathbf{R})$ the k-direct sum Hilbert space with scalar product (3.4). Let $\underline{a} = (\alpha_1, \dots, \alpha_k) \in \mathbf{R}^k_+$, $\underline{b} = (\beta_1, \dots, \beta_k) \in \mathbf{R}^k_+$ and $\underline{g} = g^1 \oplus \cdots \oplus g^k \in K$ be given.*

I. *Suppose the coherent k-tuple $(\underline{g}; \underline{\alpha}, \underline{\beta})$ is a Weyl-Heisenberg multi frame as well as either orthogonal or super-coherent (or both). Then the standard dual of the frame $\mathcal{WH}_{\underline{g};\underline{\alpha},\underline{\beta}}$ in K is coherent and of the form $\mathcal{WH}_{\underline{\tilde{g}};\underline{\alpha},\underline{\beta}}$ for some $\underline{\tilde{g}} \in K$ and therefore the k-tuple $(\underline{\tilde{g}}; \underline{\alpha}, \underline{\beta})$ is a Weyl-Heisenberg multi frame dual to $(\underline{g}; \underline{\alpha}, \underline{\beta})$. Moreover, the associated tight frame to $(\underline{g}; \underline{\alpha}, \underline{\beta})$ in K is coherent as well and of the form $(\underline{g}^\#; \underline{\alpha}, \underline{\beta})$ for some $\underline{g}^\# \in K$.*

II. *Suppose the coherent k-tuple $(\underline{g}; \underline{\alpha}, \underline{\beta})$ is a Weyl-Heisenberg multi Riesz basis as well as either orthogonal or ultra-coherent (or both). Then the standard dual of the Riesz basis for its span $\mathcal{R}_{\underline{g};\underline{\alpha},\underline{\beta}}$ is of the form $\mathcal{R}_{\underline{\tilde{g}};\underline{\alpha},\underline{\beta}}$ for some $\underline{\tilde{g}} \in K$ and therefore the k-tuple $(\underline{\tilde{g}}; \underline{\alpha}, \underline{\beta})$ is a Weyl-Heisenberg multi Riesz basis, dual to $(\underline{g}; \underline{\alpha}, \underline{\beta})$. Furthermore, the associated orthonormal set of $(\underline{g}; \underline{\alpha}, \underline{\beta})$ is as well of the form $(\underline{g}^\#; \underline{\alpha}, \underline{\beta})$ for some $\underline{g}^\# \in K$.*

During the proof of this result we shall need the following lemma:

LEMMA 3.8. *Suppose $\underline{g} \in K$ and $\underline{\alpha}, \underline{\beta} \in \mathbf{R}^k_+$ with $(\underline{g}; \underline{\alpha}, \underline{\beta})$ an ultra-coherent set. Then:*

1. *$(\underline{g}; \underline{\alpha}, \underline{\beta})$ is a Weyl-Heisenberg multi frame if and only if $(\underline{g}; \frac{1}{\underline{\beta}}, \frac{1}{\underline{\alpha}})$ is a Weyl-Heisenberg multi Riesz basis (where $\frac{1}{\underline{\alpha}} = (\frac{1}{\alpha_1}, \dots, \frac{1}{\alpha_k})$, $\frac{1}{\underline{\beta}} = (\frac{1}{\beta_1}, \dots, \frac{1}{\beta_k})$);*
2. *Suppose now $(\underline{g}; \underline{\alpha}, \underline{\beta})$ is a Weyl-Heisenberg multi frame. Then:*
   (a) *$(\underline{g}'; \underline{\alpha}, \underline{\beta})$ is a dual of $(\underline{g}; \underline{\alpha}, \underline{\beta})$ if and only if $(\frac{1}{\underline{\alpha}\underline{\beta}}\underline{g}'; \frac{1}{\underline{\beta}}, \frac{1}{\underline{\alpha}})$ is a dual multi Riesz basis of $(\underline{g}; \frac{1}{\underline{\beta}}, \frac{1}{\underline{\alpha}})$;*

(b) $(\underline{\tilde{g}};\underline{\alpha},\underline{\beta})$ *is the standard dual of* $(\underline{g};\underline{\alpha},\underline{\beta})$ *if and only if* $(\frac{1}{\underline{\alpha\beta}}\underline{\tilde{g}};\frac{1}{\underline{\beta}},\frac{1}{\underline{\alpha}})$ *is the standard dual of* $(\underline{g};\frac{1}{\underline{\beta}},\frac{1}{\underline{\alpha}})$;

(c) $(\underline{g^{\#}};\underline{\alpha},\underline{\beta})$ *is the associated tight multi frame of* $(\underline{g};\underline{\alpha},\underline{\beta})$ *if and only if* $(\frac{1}{\sqrt{\underline{\alpha\beta}}}\underline{g^{\#}};\frac{1}{\underline{\beta}},\frac{1}{\underline{\alpha}})$ *is the associated orthonormal multi set of* $(\underline{g};\frac{1}{\underline{\beta}},\frac{1}{\underline{\alpha}})$;

Proof of Lemma 3.8. 1. The claim follows directly from 2(b) below, since the boundedness of the frame operator of the multiframe is equivalent to the boundedness of the frame operator of the multi Riesz basis.

2. (a) If $(\underline{g}^{'};\underline{\alpha},\underline{\beta})$ is a dual multiframe of $(\underline{g};\underline{\alpha},\underline{\beta})$, then by Theorem 2.13 $(g^{'j};\alpha_j,\beta_j)$ is orthogonal to $(g^l;\alpha_l,\beta_l)$ for $l\neq j$ and $(g^{'j};\alpha_j,\beta_j)$ is a dual of $(g^j;\alpha_j,\beta_j)$. Using Theorem 3.6 we get immediately $(g^{'j};\frac{1}{\beta_j},\frac{1}{\alpha_j})$ is orthogonal to $(g^l;\frac{1}{\beta_l},\frac{1}{\alpha_l})$ for $j\neq l$, and $(\frac{1}{\alpha_j\beta_j}g^{'j};\frac{1}{\beta_j},\frac{1}{\alpha_j})$ is a biorthogonal basis of $(g^j;\frac{1}{\beta_j},\frac{1}{\alpha_j})$. Note $\alpha_j=\alpha_1$ and $\beta_j=\beta_1$ and, for fulfilling the norm condition, $\underline{g}'$ should be divided by $k\alpha_1\beta_1=\underline{\alpha\beta}$. Thus the duality property for the multi Riesz bases is proved. The converse proof follows a similar approach.

(b) The previous proof showed $\underline{g}'$ is a dual of the other one. The remaining issue is to prove the standard duality is preserved. Let us assume $(\underline{g}';\underline{\alpha\beta})$ is the standard dual multiframe of $(\underline{g};\underline{\alpha},\underline{\beta})$. Then we have to prove that $(\frac{1}{\underline{\alpha\beta}}\underline{g}';\frac{1}{\underline{\beta}},\frac{1}{\underline{\alpha}})$ is the standard biorthogonal of $(\underline{g};\frac{1}{\underline{\beta}},\frac{1}{\underline{\alpha}})$. This claim comes as a conclusion of norm minimality property similar to the one-generator case. Here are the details of this program. Let $(\underline{g}^d;\underline{\alpha},\underline{\beta})$ be a dual multiframe of $(\underline{g};\underline{\alpha},\underline{\beta})$ and $\underline{g}^{'};\underline{\alpha},\underline{\beta})$ be its standard dual multiframe. Set $\underline{h}=\underline{g}^d-\underline{g}^{'}$ and consider $(\underline{h};\underline{\alpha},\underline{\beta})$. Applying Theorem 2.13, d), we obtain $(h_l;\alpha_l,\beta_l)$ is orthogonal to $(g_l^{'};\alpha_l,\beta_l)$ in $l^2(\mathbf{Z}^2)$ sense. Applying Wexler-Raz identity (3.3), we obtain $(h_l;\frac{1}{\beta_l},\frac{1}{\alpha_l})$ is orthogonal to $(g_l^{'};\frac{1}{\beta_l},\frac{1}{\alpha_l})$ in $L^2(\mathbf{R})$ sense. Consequently, $\|\underline{g}^d\|^2=\|\underline{g}^{'}\|^2+\|\underline{h}\|^2\geq\|\underline{g}^{'}\|^2$. Therefore among all dual multiframe generators $\underline{g}^d$, the standard dual multi frame $\underline{g}^{'}$ has the minimum K-norm. Now for multi Riesz basis $(\underline{g};\underline{\alpha},\underline{\beta})$, the equivalent statement is obvious. Indeed, if $(\underline{g}^{'};\underline{\alpha},\underline{\beta})$ is the standard dual, and $(\underline{g}^o;\underline{\alpha},\underline{\beta})$ is any other biorthogonal multi Riesz basis then $(\underline{g}^o-\underline{g}^{'};\underline{\alpha},\underline{\beta})$ is orthogonal to $(\underline{g}^{'};\underline{\alpha},\underline{\beta})$ in the $L^2(\mathbf{R})$ sense, hence $\|\underline{g}^o\|\geq\|\underline{g}^{'}\|$. Once this norm minimality is obtained, the statement (b) is straightforwardly obtained.

(c) An easy computation using Theorem 3.6 shows that $(\underline{g^{\#}};\underline{\alpha},\underline{\beta})$ is a tight multiframe with bound 1, if and only if $(\frac{1}{\sqrt{\underline{\alpha\beta}}}\underline{g^{\#}};\frac{1}{\underline{\beta}},\frac{1}{\underline{\alpha}})$ is an orthonormal multi set.

Next, $\underline{g^{\#}}$ is the associated tight multiframe generator if and only if $\frac{1}{\sqrt{\underline{\alpha\beta}}}\underline{g^{\#}}$ is the associated orthonormal multiset is proved by extending the GNS construction to a representation of the WH group on K, similar to the one done in [**DLL95**]. $\square$

Proof of Theorem 3.7. I. The result comes from the commutation relation $SU(m\underline{\alpha},n\underline{\beta})=U(m\underline{\alpha},n\underline{\beta})S$ where S is the frame operator associated to $\mathcal{WH}_{\underline{g};\underline{\alpha},\underline{\beta}}$ in K. In the orthogonal case, S is a direct sum of frame operators $S_j:H\rightarrow H$, $S=S_1\oplus\cdots\oplus S_k$ and for each j, $S_jU(m\alpha_j,n\beta_j)=U(m\alpha_j,n\beta_j)S_j$ (Theorem 3.1, I.c). In the super-coherent case, the computation goes similarly to the scalar case (see [**Balan97**]).

II. In the orthogonal case, the grammian operator G decomposes into a direct sum of k grammian operators associated to each individual Riesz basis for its span,

$G_j : l^2(\mathbf{Z}^2) \to L^2(\mathbf{R})$, $G = G_1 \oplus \ldots \oplus G_k : l^2(\mathbf{Z}^2) \oplus \ldots \oplus l^2(\mathbf{Z}^2) \to L^2(\mathbf{R})$. Then $G^{-1} = G_1^{-1} \oplus \ldots \oplus G_k^{-1}$ as well as $G^{-1/2} = G_1^{-1/2} \oplus \ldots \oplus G_k^{-1/2}$ and the result comes from Theorem 3.1, II.d. The ultra-coherent case comes from from the lemma 3.8 by constructing the adjoint multiframe.

□

REMARK 3.9. An alternative proof of Theorem 3.6, and Lemma 3.8, can be obtained by considering the GNS construction associated to von Neumann algebra $\{U(m\underline{\alpha}, n\underline{\beta})\ ;\ m,n \in \mathbf{Z}\}''$.

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