Bi-Lipschitz Euclidean Embeddings of Metric Spaces induced by Finite Group Representations

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Preprints

Preprints:

1. R.B., Naveed Haghani, Maneesh Singh, “Permutation-Invariant Representations with Applications to Graph Deep Learning”, ACHA, vol. 9 (2025)
2. R.B., Efstratios Tsoukanis, “Relationships between the Phase Retrieval Problem and Permutation Invariant Embeddings”, arXiv:2306.13111 [math.FA] , [cs.IT] , [math.IT]
3. R.B., Efstratios Tsoukanis, “G-Invariant Representations using Coorbits: Bi-Lipschitz Properties”, arXiv:2308.11784 [math.RT]
4. R.B., Efstratios Tsoukanis, “G-Invariant Representations using Coorbits: Injectivity Properties”, arXiv:2310.16365 [math.RT]
5. R.B, Efstratios Tsoukanis, Matthias Wellershoff, “Stability of sorting based embeddings”, arXiv:2410.05446 [math.FA]
6. N. Dym, M. Wellershoff, E. Tsoukanis, D. Levy, R. Balan, “Quantitative Bounds for Sorting-based Permutation-Invariant Embeddings”, arXiv:2510.22186[cs.LG, cs.IT, math.FA, math.MG]

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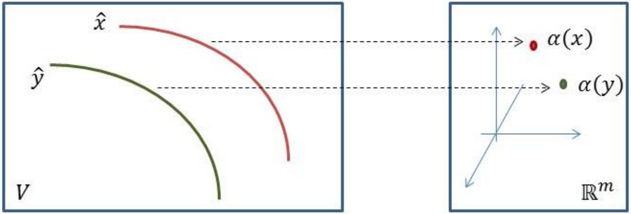
High-Level View

In this talk, we discuss Euclidean embeddings of metric spaces induced by orthogonal representations of finite groups G acting on a linear space V with inner product.

Problem: Construct bi-Lipschitz embeddings of the metric space $\hat{V} = V/\sim$ of orbits, $\alpha : \hat{V} \to \mathbb{R}^m$, where $\mathbf{d}([x],[y]) = \inf_{u\in[x],v\in[y]} \|u-v\|$

$$a_0 \mathbf{d}([x],[y]) \leq \|\alpha([x]) - \alpha([y])\|_2 \leq b_0 \mathbf{d}([x],[y]).$$



The Program

Given a discrete group G acting unitarily on a normed real space V, we formulate four general problems

1. Construct injective embeddings of the quotient space V/G, $\alpha : \hat{V} \to \mathbb{R}^m$. The injectivity problem.
2. Construct/Obtain bi-Lipschitz properties for the Euclidean embedding $\alpha : \hat{V} \to \mathbb{R}^m$. The stability problem.
3. Develop algorithms for inversion $\alpha^{-1} : \mathbb{R}^m \to \hat{V}$. The recovery problem.
4. Analyze specific cases. Applications.

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Today we discuss results about the first two problems: injectivity,

bi-Lipschitz stability.

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I. Phase Retrieval Problems

Since 2006 ACHA paper[1] lots of research on this theme.
The group: $G = O(1) = \{+1, -1\}$ or $G = U(1) \sim \mathbb{T}^1$ acting on $\mathbb{K}^n$.
Embedding: $x \mapsto \{|\langle x, f_k \rangle|\}_{k \in [m]}$ for a fixed frame $\{f_1, \cdots, f_m\} \subset \mathbb{K}^n$, $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$.
Two type of results of particular interest:

- Minimal embeddings: For $\mathbb{K} = \mathbb{R}$, $m_{min} \geq 2n - 1$ [1]; For $\mathbb{K} = \mathbb{C}$: $m_{min} \leq 4n - 4$; $m_{min} = 4n - 4$ when $n = 2^p + 1$ [Conca,Edidin,Hering, Vinzant'15]; [Vinzant'15]: $n = 4$, $m_{min} = 11 = 4n - 5$
- Bi-Lipschitz: [EldarMend'14,BandCahlMixnNels'14,BWang'15,BZou'15'16] Any finite-dimensional injective embedding is bi-Lipschitz. Global inverse Lipschitz.

[1] R.B, Pete Casazza, Dan Edidin, On Signal Reconstruction without Noisy Phase, Appl. Comp. Harm. Anal., 20 (2006)

II. Graph Learning Problems

Given a data graph (e.g., social network, transportation network, citation network, chemical network, protein network, biological networks):

- Graph adjacency or weight matrix, $A \in \mathbb{R}^{n \times n}$;
- Data matrix, $X \in \mathbb{R}^{n \times r}$, where each row corresponds to a feature vector per node.

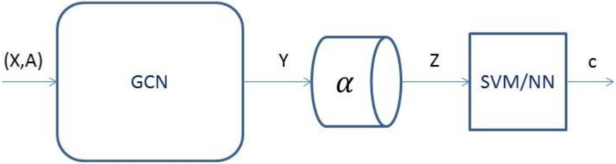
Contruct a map $f : (A, X) \to f(A, X)$ that performs:

1. classification: $f(A, X) \in \{1, 2, \cdots, c\}$
2. regression/prediction: $f(A, X) \in \mathbb{R}$.

Key observation: The outcome should be invariant to vertex permutation: $f(PAP^T, PX) = f(A, X)$, for every $P \in \mathcal{S}_n$.

Graph Deep Learning with GCN/GNN

Our approach for these learning tasks (classification or regression) is based on the following scheme (see GCN[2] and equivariance[3]):



where α is a permutation invariant map (embedding), and SVM/NN is a single-layer or a deep neural network (Support Vector Machine or a Fully Connected Neural Network) trained on invariant representations.

Our focus is on the α component.

[2]Kipf, T. N. and Welling, M., Semi-Supervised Classification with Graph Convolutional Networks, arXiv e-prints , arXiv:1609.02907 (Sep 2016).

[3]H. Maron, E. Fetaya, N. Segol, Y. Lipman, On the Universality of Invariant Networks, arXiv:1901.09342 [cs.LG] (May 2019).

III. Assignment Problems

The Graph Isomorphism Problem

Consider two graphs $G = (\mathcal{V}, \mathcal{E})$ and $\tilde{G} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}})$ with n nodes. The graph isomorphism problem is the computational problem of determining whether these graphs are identical after a relabeling of nodes.

If A and $\tilde{A}$ denote their adjacency matrices, these graphs are isomorphic if and only if $\tilde{A} = \Pi A \Pi^T$ for some permutation matrix $\Pi \in \mathcal{S}_n$.

Current state-of-the-art (Wikipedia): Babai (2015,2017) presented a quasi-polynomial algorithm with running time $2^{O((\log n)^c)}$, for some fixed $c > 0$. Helfgott (2017) claims that one can take $c = 3$.

Similar problem can be stated for weighted graphs: $A, \tilde{A} \in \mathrm{Sym}(n)$ with nonnegative entries, isomorphic if and only if $\tilde{A} = \Pi A \Pi^T$ for some $\Pi \in \mathcal{S}_n$.

Graph Alignment Problems

Consider two $n \times n$ symmetric matrices A, B. The "vanilla" alignment problem for quadratic forms asks for the orthogonal matrix $U \in O(n)$ that minimizes

$$\|UAU^T - B\|_F^2 := trace((UAU^T - B)^2) = \|A\|_F^2 + \|B\|_F^2 - 2trace(UAU^T B).$$

The solution is well-known and depends on the eigendecomposition of matrices A, B: if $A = U_1 D_1 U_1^T$, $B = U_2 D_2 U_2^T$ then

$$U_{opt} = U_2 U_1^T \quad , \quad \|U_{opt} A U_{opt}^T - B\|_F^2 = \sum_{k=1}^{n} |\lambda_k - \mu_k|^2,$$

where $D_1 = diag(\lambda_k)$ and $D_2 = diag(\mu_k)$ are diagonal matrices with eigenvalues ordered monotonically.

Quadratic Assignment Problem (QAP)

The challenging case is when U is constrained to the permutation group as is the case in the *graph matching problem*. In this case, the optimization problem becomes

$$\min_{U \in \mathcal{S}_n} \|UAU^T - B\|_F$$

which turns into a QAP: $\max_{U \in \mathcal{S}_n} trace(UAU^T B)$.
This is equivalent to computing the natural distance $d(\hat{A}, \hat{B}) = \min_{P,Q \in \mathcal{S}_n} \|PAP^T - QBQ^T\|_F$ between the equivalence classes $\hat{A}, \hat{B} \in \widehat{\mathrm{Sym}(n)}$ induced by action $(\Pi, A) \mapsto \Pi A \Pi^T$.
How is this connected to the embedding problem? If one can design an *efficient* nearly isometric map $\Phi : Sym(n) \to \mathbb{R}^m$ so that
(1) $\Phi(PAP^T) = \Phi(A)$ for all $P \in \mathcal{S}_n$ and $A \in Sym(n)$, and

(2) $$(1-\delta) \min_{P \in \mathcal{S}_n} \|PAP^T - B\| \le \|\Phi(A) - \Phi(B)\| \le (1+\delta) \min_{P \in \mathcal{S}_n} \|PAP^T - B\|,$$

then the QAP solved efficiently up to a multiplicative factor.

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Problem Setup

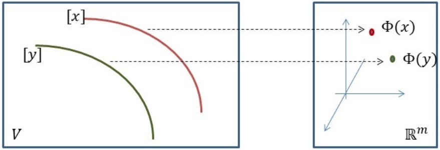
Consider a group $G \subset O(d)$ acting on the Euclidean space $V = \mathbb{R}^d$.

General problem

Construct an embedding map $\Phi : V \to \mathbb{R}^m$

1. Invariance: $\Phi(U_g x) = \phi(x)\ \forall g \in G, x \in V$
2. Injectivity: if $\Phi(x) = \Phi(y)$ then there exists $g \in G$ so that $y = U_g x$.
3. Φ is bi-Lipschitz on $(\hat{V} = V/G, \mathbf{d})$:

$$a_0 \inf_{u\in[x],v\in[y]} \|u - v\| \leq \|\Phi(x) - \Phi(y)\| \leq b_0 \inf_{u\in[x],v\in[y]} \|u - v\|.$$



Approaches

Over the past many years, several constructions have been proposed:

1. Invariant Polynomials: Hilbert, Noether, ..., Cahill[4], Bandeira[5]
2. Kernels: replace monomials by other kernels, e.g. $e^{i\omega x}$, e^{-x^2}, $\sigma(\langle x, a\rangle)$[6]
3. Sorting: extends the 1-D sorting, $x \mapsto \downarrow x$ [7,8]

1+2: *sum pooling* layer; 3: *max pooling* layer deep nets[9], [10].

[4] J. Cahill, A. Contreras, A.C. Hip, Complete Set of translation Invariant Measurements with Lipschitz Bounds, Appl. Comput. Harm. Anal. 49 (2020), 521–539.

[5] A. Bandeira, B. Blum-Smith, J. Kileel, J. Niles-Weed, A. Perry, A.S. Wein, Estimation under group actions: Recovering orbits from invariants, ACHA 66 (2023)

[6] D. Yarotsky, Universal approximations of invariant maps by neural networks, Constructive Approximation (2021)

[7] R. Balan, N. Haghani, M.Singh, Permutation-Invariant Representations with Applications to Graph Deep Learning, arXiv:2203.07546

[8] J. Cahill, J.W. Iverson, D.G. Mixon, D. Packer, Group-invariant max filtering, arXiv:2205.14039.

[9] O. Vinyals, S. Bengio, M. Kudlur, Order Matters: Sequence to sequence for sets, ICLR 2016

[10] H. Maron, H. Ben-Hamu, N. Shamir, Y. Lipman, Invariant and equivariant graph networks, ICLR 2019

Idea

Consider the special case $G = S_n$ is the symmetric group acting by permutation matrices on $V = \mathbb{R}^n$.

The ring of invariant polynomials is generated by the elementary symmetric polynomials $e_1, ..., e_n$, $\mathbb{R}[X_1, \cdots, X_n]^{S_n} \simeq \mathbb{R}[e_1, \cdots, e_n]$. There is a natural embedding $\mathbb{R}^n / S_n \hookrightarrow \mathbb{R}^n$, $x \mapsto (e_1(x), \cdots, e_n(x))$.
Drawback: it is not bi-Lipschitz.

Idea

Consider the special case $G = S_n$ is the symmetric group acting by permutation matrices on $V = \mathbb{R}^n$.

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Drawback: it is not bi-Lipschitz.

Alternatively: Consider the embedding $\downarrow: \mathbb{R}^n/S_n \hookrightarrow \mathbb{R}^n$, $x \mapsto \downarrow(x)$, that sorts monotone decreasing the vector x.
Key obervation: $\min_{P \in S_n} \|x - Py\|_2 = \|\downarrow x - \downarrow y\|_2$.
Hence: $\downarrow$ is an *isometric* embedding of $\mathbb{R}^n/S_n$ into $\mathbb{R}^n$,

Sorting based Representations and G-invariance

Assume V is a real d-dimensional Hilbert space and G a finite orthogonal group of size $N=|G|$, acting on V, $\{U_g, g\in G\}$.
Fix a generator $w\in V$ (call it, *window*, or *template*, or *wavelet*) and consider the nonlinear map induced by sorting its coorbit:

$$\phi_w : V \rightarrow \mathbb{R}^N \quad , \quad \phi_w(x) = \downarrow ((\langle x, U_g w\rangle)_{g\in G}) \,.$$

where $\downarrow(y)=(y_{\pi(i)})_{i\in[N]}$ is the non-increasing sorting operator: $y_{\pi(1)}\geq\cdots\geq y_{\pi(N)}$.
Key observations:

1. $\phi_w(U_g x)=\phi_w(x)$, ϕ is G-invariant.
2. ϕ_w is piecewise linear (in fact, $\phi_w(x)=\phi_x(w)$, and $(w,x)\mapsto\phi_w(x)$ is piecewise bilinear).

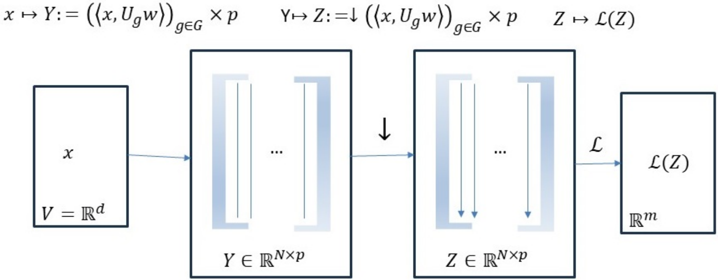
G-Invariant Coorbit Representations

For a collection $\mathbf{w} = (w_1, \cdots, w_p) \in V^p$ the sorted coorbit representation:

$$\Phi_{\mathbf{w}} : V \to \mathbb{R}^{N \times p} \quad , \quad \Phi_{\mathbf{w}}(x) = \left[\phi_{w_1}(x) | \cdots | \phi_{w_p}(x)\right].$$

Apply a dimension-reduction linear map $\mathcal{L} : \mathbb{R}^{N \times p} \to \mathbb{R}^m$, the G-invariant coorbit representation:

$$\Psi_{\mathbf{w},\mathcal{L}} : V \to \mathbb{R}^m \quad , \quad \Psi_{\mathbf{w},\mathcal{L}}(x) = \mathcal{L}(\Phi_{\mathbf{w}}(x))$$



In particular, if $S \subset [N] \times [p]$, $\Phi_{\mathbf{w},S} := \Psi_{\mathbf{w},1_S} = \Phi_{\mathbf{w}}|_S$.

G-Invariant Coorbit Representations

Special cases:

1. For $G = S_n$ and $V = \mathbb{R}^{n\times d}$ with action $(P, X) \mapsto PX$ [11] introduced the embedding $\beta_A(X) = \downarrow(XA)$, for *key* $A \in \mathbb{R}^{d\times D}$ and sorting operator acting independently in each column. This is of the type $\Psi_{\mathbf{w},\mathcal{L}}$ for $w_1 = \delta_1 \cdot a_1^T, \ldots$, $w_D = \delta_1 \cdot a_D^T$, where $\delta_1 = (1, 0, \cdots, 0)^T$ and $A = [a_1 | \cdots | a_D]$, and $\mathcal{L}$ a restriction operator to an appropriate subset $S \subset [n!] \times [D]$ of size nD.

2. The *max filter* introduced in [12] for some template $w \in V$ is defined by $\langle\langle \cdot, w\rangle\rangle : V \to \mathbb{R}$, $\langle\langle x, w\rangle\rangle = \max_{g\in G} \langle x, U_g w\rangle$. Equivalent recasting: $\langle\langle x, w\rangle\rangle = \mathcal{L}(\Phi_w(X))$, for a restriction operator $\mathcal{L}$ to the subset $S = \{1\}$.

3. The operator $\Psi_{\mathbf{w},\mathcal{L}}$, $\Psi_{\mathbf{w},\mathcal{L}}(X) = \mathcal{L}(\Phi_{\mathbf{w}}(X))$ has been introduced in [13]

[11] R. Balan, N. Haghani, M.Singh, Permutation Invariant Representations with Applications to Graph Deep Learning, arXiv:2203.07546 (2022)

[12] J. Cahill, J. W. Iverson, D. G. Mixon, D. Packer, Group-invariant max filtering, arXiv:2205.14039 (2022)

[13] R.B, Efstratios Tsoukanis, Matthias Wellershoff, “Stability of sorting based embeddings”, arXiv:2410.05446 (2024)

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Main Results

Injectivity

Let $V_G = \{x \in V : U_g x = x \,,\, \forall g \in G\}$, $d_G = dim(V_G)$, $q \geq 0$ and for $g = (g_1, \cdots, g_n), h = (h_1, \cdots, h_n) \in H_n \subset G^n$ distinct, $\rho_n(q) = \max_{g,h} \gamma^q_{g,h}$ where

$$\gamma^q_{g,h} = semi.alg.dim. \{(x, y) \in V \times V : dim(span\{U_{g_k}x - U_{h_k}y, k \in [n]\}) = q\}$$

Theorem (R.B.,E.Tsoukanis '23-'25)

In any of the following cases

1. *Assume* $p \geq 2\,dim(V) - d_G$ *and set* $\mathbf{n} = (k_1, \cdots, k_p) \in [N]^p$.
2. *Fix* $n \in [N]$ *and choose* $p > \max_{q \in [n]} \frac{1}{q}(\rho_n(q) - d_G - 1)$. *Set* $\mathbf{n} = (n, \cdots, n) \in [N]^p$.
3. *Choose* $p \geq 1$ *and* $\mathbf{n} = (n_1, \cdots, n_p) \in [N]^p$ *so that* $\max_{q_1 \in [n_1], \cdots, q_p \in [n_p]} \left(min_{i \in [p]} \rho_{n_i}(q_i) - (q_1 + \cdots + q_p)\right) \leq d_G$.

For a generic (w.r.t. Zariski topology) $\mathbf{w}$ *and for any* $S \subset [N] \times [p]$ *with* $|\{k : (k, i) \in S\}| \geq n_i$, *the map* $\Phi_{\mathbf{w},S} : (\widehat{V}, \mathbf{d}) \to (\mathbb{R}^{|S|}, \|\cdot\|_2)$ *is injective.*

Main Results (2)

Theorem (R.B, E.T., M. Wellershoff '24)

Consider the same setup as before. Assume $\mathbf{w} \in V^p$ *and* $\mathcal{L} : \mathbb{R}^{N\times p} \to \mathbb{R}^m$ *so that* $\Psi_{\mathbf{w},\mathcal{L}} : (\widehat{V}, \mathbf{d}) \to (\mathbb{R}^m, \|\cdot\|_2)$ *is* **injective**.

1. *Themap* $\Psi_{\mathbf{w},\mathcal{L}} : (\widehat{V}, \mathbf{d}) \to (\mathbb{R}^m, \|\cdot\|_2)$ *is bi-Lipschitz. Let* a_0, b_0 *denote its bi-Lipschitz constants.*
2. *If* $f : V \to H$ *is a Lipschitz continuous function so that* $f(U_g x) = f(x)$ *for all* g, x*, where* H *is a Hilbert space, then there exists a Lipschitz continuous function* $g : \mathbb{R}^m \to H$ *so that* $f = g \circ \Psi_{\mathbf{w},\mathcal{L}}$*, i.e.* $f(x) = g(\Psi_{\mathbf{w},\mathcal{L}}(x))$*. Furthermore,* $Lip(g) \leq Lip(f)/a_0$*.*
3. *Assume* $g : \mathbb{R}^m \to H$ *is a Lipschitz function with Lipschitz constant* $Lip(g)$*. Then* $f = g \circ \Psi_{\mathbf{w},\mathcal{L}} : V \to H$ *is G-invariant and Lipschitz, with Lipschitz constant* $Lip(f) \leq b_0 Lip(g)$*.*

Its proof is based on Kirszbraun's extension theorem.

Existing Results

Injectivity problem

Over the past 15 years or so, there have been works that recognized the difference between *generating polynomials* and *separating invariants*[14]

A seminal paper that resurfaces results on semi-algebraic sets is [15]. The method goes back to earlier works in phase retrieval[16].

More recently, in the context of G-invariance, [17],[18], or permutation invariance[19]

[14] Emilie Dufresne, Separating invariants and finite reflection groups, Advances in Mathematics 221 (2009), no. 6, 1979–1989.

[15] Dym Nadav, Steven J. Gortler. "Low dimensional invariant embeddings for universal geometric learning." arXiv preprint arXiv:2205.02956.

[16] R. Balan, P. Casazza, D. Edidin, On signal reconstruction without phase, ACHA 20(2006)

[17] D. G. Mixon, D. Packer, Max filtering with reflection groups, arXiv:2212.05104

[18] R. Balan, E. Tsoukanis, G-invariant representations using coorbits: Injectivity properties, arXiv:2310.16365

[19] On the equivalence between graph isomorphism testing and function approximation with GNNs, Z. Chen, S. Villar, L. Chen, J. Bruna, NeurIPS 2019.

Existing Results (2)

Lipschitz and Bi-Lipschitz properties

Earlier results obtain Lipschitz/bi-Lipschitz properties on compacts, or certain classes of functions.

Global L/bi-L are harder to establish and typically rule out polynomial based embeddings.

So far only sorting based embeddings showed such global properties [20],[21],[22]

[20] R. Balan, E. Tsoukanis, G-invariant representations using coorbits: Bi-lipschitz properties, arXiv:2308.11784

[21] J. Cahill, J. W. Iverson, D. G. Mixon, Bilipschigz group invariants, arXiv:2305.17241

[22] D. G. Mixon, Y. Qaddura, Injectivity, stability, and positive definiteness of max filtering, arXiv:2212.11156

Sketch of Proof: Injectivity Result

Define the *"bad"* set of **w**'s that fail to separate all distinct classes:

$$\mathcal{F} = \{\mathbf{w} \in V^p \ , \ \exists x \not\sim y \ \Phi_{\mathbf{w}}(x) = \Phi_{\mathbf{w}}(y) \ \}.$$

The work is to embed $\mathcal{F}$ into a semi-algebraic set of semi-algebraic dimension strictly less than $pd = p\,dim(V)$.
This technique is called "lift-and-project'[23]: we construct a semi-algebraic vector bundle embedded into a certain Grassmanian vector bundle $\gamma_{n,k}^{\perp}$.
The bad set $\mathcal{F}$ is then indentified with a subset of the projection of this vector bundle into its second component.
The full result for $\Psi_{\mathbf{w},\mathcal{L}}$ follows from analyzing the semi-algebraic dimension of the difference-set $\{\Phi_{\mathbf{w}}(x) - \Phi_{\mathbf{w}}(y)\}$.

[23] R. Balan, P. Casazza, D. Edidin, On signal reconstruction without phase, ACHA 20(2006)

Sketch of Proof: Lower Lipschitz bound

The proof is by contradiction. Consider the simpler case when $\mathcal{L}$ is given by restriction to a subset $S \subset [N] \times [p]$.

1. If lower Lipschitz constant vanishes, then it must vanish locally: there are $(x_n)_n, (y_n)_n$ such that

$$\lim_{n\to\infty} \frac{\|\Phi_{\mathbf{w},S}(x_n) - \Phi_{\mathbf{w},S}(y_n)\|^2}{\mathbf{d}([x_n],[y_n])^2} = 0$$

and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = z_1, \ \|x_n\| = 1, \ \|y_n\| \le 1, \ \|z_1\| = 1$$

and they are aligned with one another:

$$\|x_n - y_n\| = \min_{g\in G} \|x_n - U_g y_n\| \tag{4.1}$$

$$\|x_n - z_1\| = \min_{g\in G} \|x_n - U_g z_1\| \tag{4.2}$$

$$\|y_n - z_1\| = \min_{g\in G} \|y_n - U_g z_1\| \tag{4.3}$$

Lower Lipschitz bound

2. We construct inductively $z_2, z_3, ..., z_d$ such that for all $1 \leq k \leq d-1$:

$$\|z_{k+1}\| \ll \|z_k\|, \;\; \dim(\operatorname{span}(z_1, \ldots, z_k)) = k$$

and the local lower Lipschitz constant vanishes in a convex set $\{\sum_{r=1}^{k} a_r z_r \;,\; |a_r - 1| < \epsilon\}$.
3. For $k = d$ this construction defines a non-empty open set $\{\sum_{r=1}^{k} a_r z_r \;,\; |a_r - 1| < \epsilon\}$ where the local lower Lipschitz constant vanishes.
4. Finally, we can construct $u, v \neq 0$, so that $x = u + \sum_{r=1}^{d} z_r$ and $y = v + \sum_{r=1}^{d} z_r$ satisfy $x \neq y$ and yet

$$\Phi_{\mathbf{w},S}(x) = \Phi_{\mathbf{w},S}(y).$$

This contradicts the injectivity hypothesis.

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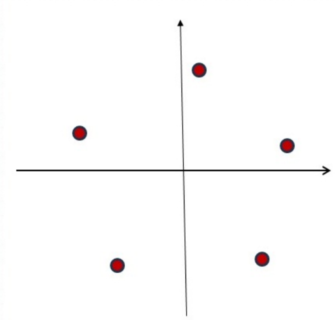
Planar Rotations

Consider the group $G =< U_{2\pi/N} >\simeq \mathbb{Z}_N$ acting on $V = \mathbb{R}^2$ by planar rotations

$$U_{2\pi/N}^k = U_{2\pi k/N} = \begin{bmatrix} cos(\frac{2\pi k}{N}) & -sin(\frac{2\pi k}{N}) \\ sin(\frac{2\pi k}{N}) & cos(\frac{2\pi k}{N}) \end{bmatrix}$$

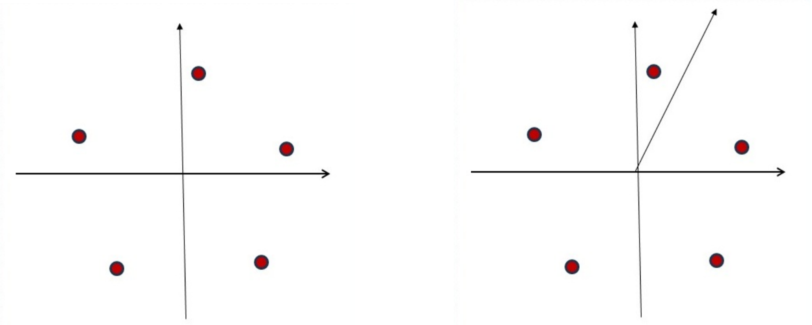
$N = 5$. A generic orbit for rotations by $\frac{2\pi}{5}$.



Planar Rotations: Metric Space

Consider the group $G =< U_{2\pi/N} >\simeq \mathbb{Z}_N$ acting on $V = \mathbb{R}^2$ by planar rotations

$$U_{2\pi/N}^k = U_{2\pi k/N} = \begin{bmatrix} cos(\frac{2\pi k}{N}) & -sin(\frac{2\pi k}{N}) \\ sin(\frac{2\pi k}{N}) & cos(\frac{2\pi k}{N}) \end{bmatrix}$$



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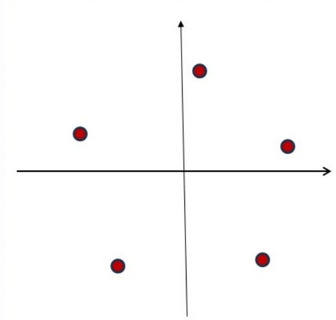

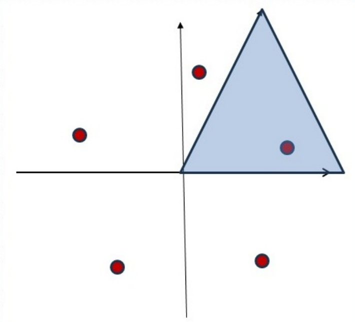

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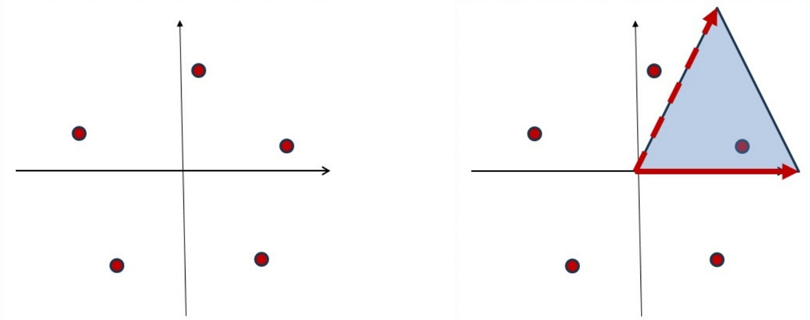

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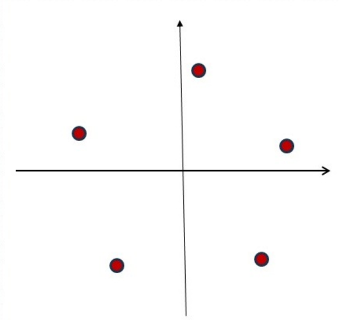
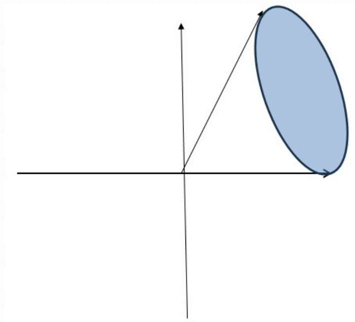


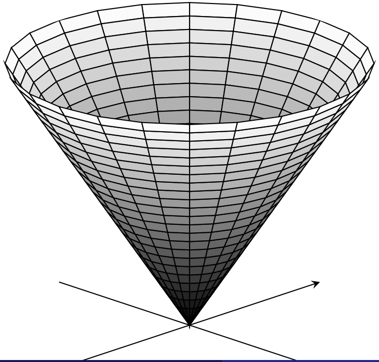


Planar Rotations: Geometric Embedding

Explicit embedding with $r = |x+iy| = \sqrt{x^2+y^2}$ and $\theta = Arg(x+iy)$:

$$\Psi : \mathbb{R}^2 \to \mathbb{R}^3 \ , \ (x,y) \mapsto \Psi(x,y) = \left(\frac{r}{N} cos(N\theta), \frac{r}{N} sin(N\theta), r\sqrt{1-\frac{1}{N^2}} \right).$$

$$\frac{1}{N\,sin(\frac{\pi}{2N})} \mathbf{d}((x_1,y_1),(x_2,y_2)) \leq \|\Psi(x_1,y_1)-\Psi(x_2,y_2)\|_2 \leq \mathbf{d}((x_1,y_1),(x_2,y_2))$$



3D embedding has distortion:

$$\frac{b_0}{a_0} = N\,sin(\frac{\pi}{2N}) \overset{N\to\infty}{\longrightarrow} \frac{\pi}{2} \approx 1.57.$$

The 2D projection:

$$(x,y) \mapsto \Psi_0(x,y) = \left(\frac{r}{N} cos(N\theta), \frac{r}{N} sin(N\theta) \right)$$

is bi-Lipschitz with distortion N.

Planar Rotations: Sorted Coorbit Embedding (1)

The following result is proved by a careful analysis of this specific case ($G =< U_a > \simeq \mathbb{Z}_N$, $a = \frac{2\pi}{N}$, $V = \mathbb{R}^2$).

Theorem (R.B, E.Tsoukanis'25)

1. *For any* $w \in \mathbb{R}^2$*, the map* $\Phi_w : \widehat{\mathbb{R}^2} \to \mathbb{R}^N$ *is never injective.*
2. *For any* $w_1, w_2 \in \mathbb{R}^2$ *and* $S = \{(q_1,1),(q_2,2)\}$ *the map* $\Phi_{\mathbf{w},S} : \widehat{\mathbb{R}^2} \to \mathbb{R}^2$ *is never injective.*
3. *Assume either one of the following holds:*
   1. $\mathbf{w} = (w_1, w_2, w_3) \in (\mathbb{R}^2)^3$ *so that* $\{U_a^{k_1} w_1, U_a^{k_2} w_2, U_a^{k_3} w_3\}$ *is a full spark frame for all integers* k_1, k_2, k_3*, and* $S = \{(1,1),(1,2),(1,3)\}$ *(the max filter);*
   2. $\mathbf{w} = (w_1, w_2) \in (\mathbb{R}^2)^2$ *so that* $\{U_{a/2}^{k_1} w_1, U_{a/2}^{k_2} w_2\}$ *is linearly independent for all* k_1, k_2 *integers, and* $S = \{(i,1),(j,1),(k,2)\}$ *(a* $\mathbf{n} = (2,1)$ *configuration) with* $i \neq j$ *and, if* N *is even then* $i + j \neq N + 1$*.*

Then generically, the map $\Phi_{\mathbf{w},S} : \widehat{\mathbb{R}^2} \to \mathbb{R}^3$ *is injective and hence*

Planar Rotations: Semi-algebraic indices

Cyclic group $< U_a > \simeq \mathbb{Z}_N$ generated by the planar rotation by $a = \frac{2\pi}{N}$.
Recall for $g, h \in G^n$,

$$\gamma_{g,h}^q = semi.alg.dim. \{(x,y) \in V \times V \ : \ dim(span\{U_{g_k}x - U_{h_k}y, k \in [n]\}) = q\}$$

$$\rho_n(q) = \max_{g,h \in H_n} \gamma_{g,h}^q$$

where $H_n = \{(g_1, \cdots, g_n) \in G^n \ , \ g_i \neq g_j, \forall i \neq j\}$.
Explicit computations:

$$\rho_1(q) = \begin{cases} 2, & q = 0, \\ 4, & q = 1, \\ -1, & q \geq 2. \end{cases} \quad \rho_2(q) = \begin{cases} 2, & q = 0, \\ 3, & q = 1 \ \& \ N \text{ odd}, \\ 4, & q = 1 \ \& \ N \text{ even}, \\ 4, & q = 2 \\ -1, & q \geq 3. \end{cases}$$

Planar Rotations: Sorted Coorbit Embedding (2)

The expressions of semi-algebraic indices imply the following result:

Theorem (R.B, E.Tsoukanis'25)

Assume N is odd. For generic $w_1, w_2, w_3 \in \mathbb{R}^2$ and every $S = \{(k_1, 1), (k_2, 1), (k_3, 2), (k_4, 3)\}$ with $k_1 \neq k_2$, the map $\Phi_{\mathbf{w},S} : \widehat{\mathbb{R}^2} \to \mathbb{R}^4$ is injective and hence bi-Lipschitz.

With additional work (replacing $H_n = \{(g_1, \cdots, g_n) \in G^n \ , \ g_i \neq g_j, \forall i \neq j\}$ with $\tilde{H}_n = \{h = (h_1, \cdots, h_n) \in H_n \ , \ \exists x \in V, \downarrow (\langle x, U_g w \rangle)_{g \in G} = (\langle x, U_{h_i} w \rangle)_{i \in [n]}\}$), it is possible to show the following result:

Theorem (R.B, E.Tsoukanis'25)

Assume N is even. For generic $w_1, w_2, w_3 \in \mathbb{R}^2$ and every $S = \{(k_1, 1), (k_2, 1), (k_3, 2), (k_4, 3)\}$ with $k_1 \neq k_2$ and $k + 1 + k_2 \neq N + 1$, the map $\Phi_{\mathbf{w},S} : \widehat{\mathbb{R}^2} \to \mathbb{R}^4$ is injective and hence bi-Lipschitz.

Thank you!

Questions?

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The Protein Dataset

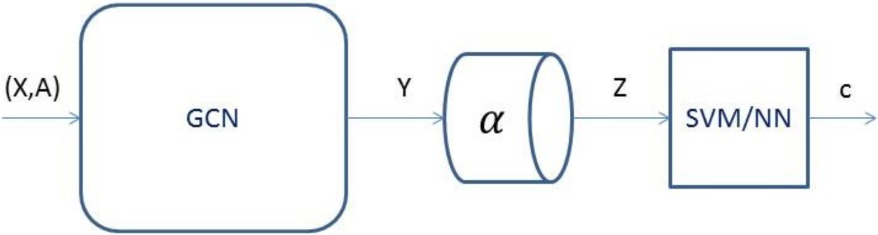
Protein Dataset: PROTEINS_FULL[24] consists of 1113 proteins: 663 non-enzymes and 450 enzymes. Each graph associated to one protein: nodes represent amino acids and edges represent the bonds between them. Number of nodes (aminoacids): varying between 20 and 620 with average of 39. Input feature vectors of size $r = 29$.
Task: the task is classification of each protein into *enzyme* or *non-enzyme*.

[24] P.D. Dobson, A.J. Doig, "Distinguishing Enzyme Structures from Non-enzymes without Alignments", J. Mol. Biol. 330, 771-783, 2003.

The Deep Network Architecture

Architecture: ReLU activation and

- GCN with $L = 3$ layers and 29 input feature vectors, and 50 hidden nodes in each layer; no dropouts, no batch normalization. output of GCN: $d = 1, 10, 50, 100$.
- Mid-layer component: α
- Fully connected NN with dense 3-layers and 150 internal units; no dropouts, with batch normalization.



The Network

Training has been done over 300 epochs with a batch size of 128. Loss function: binary cross-entropy.

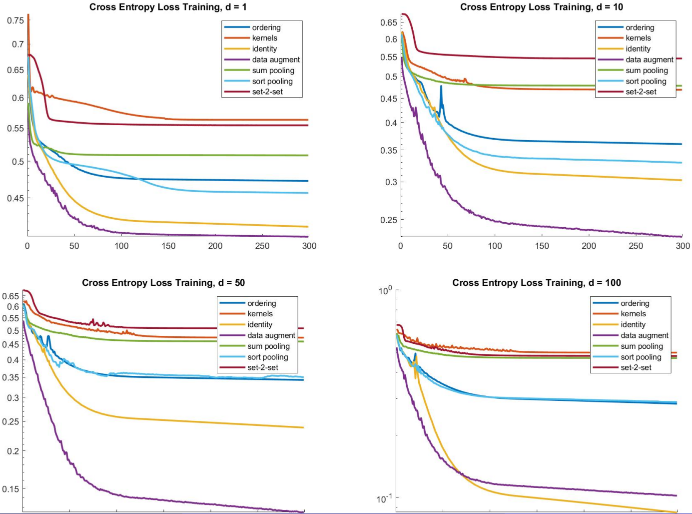
The following 7 α modules have been tested:

1. identity: $\alpha(X) = X$; no permutation invariance.
2. data augmentation: $\alpha(X) = X$ BUT the training data set has been augmented with 4 random permutatons of each graph.
3. ordering: $\alpha(X) = \downarrow(XA)$, $A = [I\ 1]$
4. kernels: $\alpha(X) = (\sum_{k=1}^{n} exp(-\|x_k - a_j\|^2))_{1 \leq j \leq m=5nd}$
5. sumpooling: $\alpha(X) = 1^T X$
6. sort-pooling: sorted by last column
7. set-to-set: introduced in [Vinyals&al.][25]

[25] Vinyals, O., Bengio, S. Kudlur, M., Order Matters: Sequence to sequence for sets, ICLR 2016.

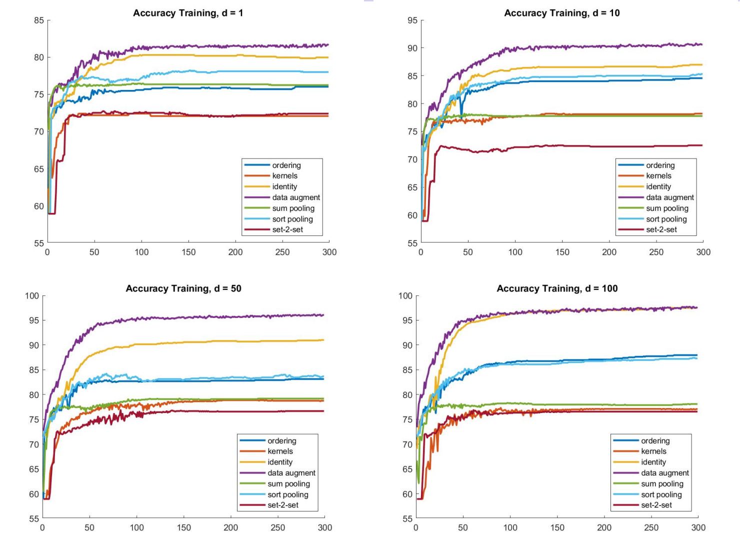
Enzyme Classification Example

Training Loss: X Entropy



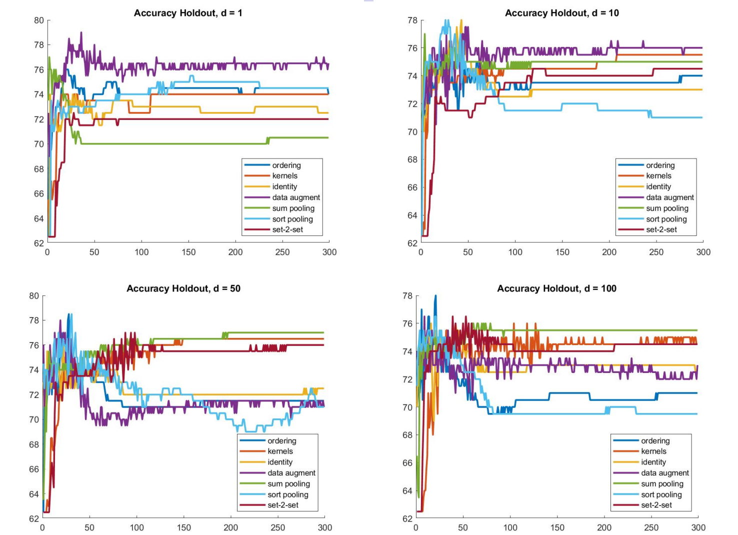
Enzyme Classification Example

Accuracy on Training set



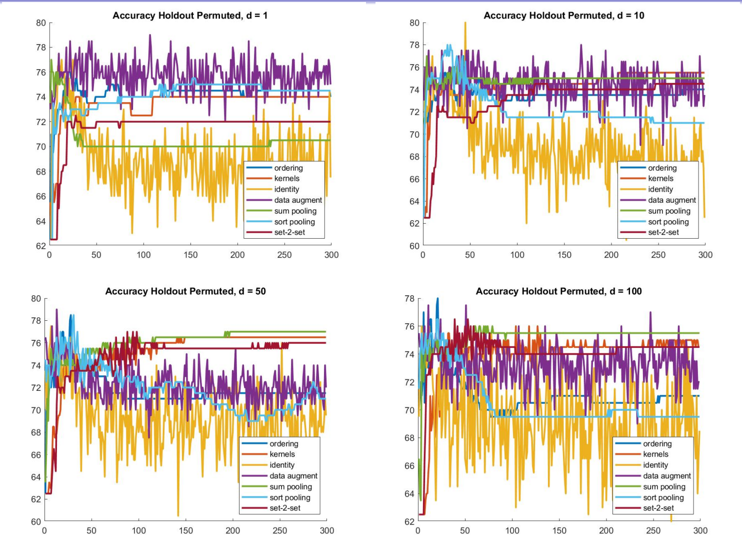
Enzyme Classification Example

Accuracy on Holdout data



Enzyme Classification Example

Accuracy on Holdout data with nodes randomly permuted



Performance Results: Accuracy

d = 50	ordering	kernels	identity	data augment	sum-pooling	sort-pooling	set-2-set
Training	83.1	78.8	91	96	79.2	83.7	76.7
Holdout	71.5	76.5	72.5	71	77	71	76
Holdout Perm	71.5	76.5	69.5	72	77	71	76

Table: Accuracy ACC(%) for enzyme/non-enzyme classification of the seven algorithms on PROTEINS_FULL dataset after 300 epochs for embedding dimension $d = 50$

For comparison: [Dobson&al.][26] obtains an accuracy of 77-80% using an SVM based classifier.

[26] P.D. Dobson, A.J. Doig, "Distinguishing Enzyme Structures from Non-enzymes without Alignments", J. Mol. Biol. 330, 771-783, 2003.

The QM9 Dataset

Dataset: QM9[27] consists of about 134,000 isomers of organic molecules made up of CHONF, each containing 10-29 atoms. see http://quantum-machine.org/datasets/ Nodes corresponds to atoms; each feature vector containins geometry (x,y,z coordinates), partial charge per atom (Mulliken charge), and atom type.

Task: the task is regression: predict a physical feature (electron energy gap $\Delta\varepsilon$) computed for each molecule.

Architecture: ReLU activation and

- GCN with $L = 3$ layers and 50 hidden nodes in each layer; no dropouts, no batch normalization; zero padding to $m = 29$ number of rows. output of GCN: $d = 1, 10, 50, 100$.
- Mid-layer component: α
- Fully connected NN with dense 3-layers and 150 internal units in each of the two hidden layers; no dropouts, with batch normalization.

[27] R. Ramakrishnan, P.O. Dral, M. Rupp, and OA. von Lilienfeld. Quantum chemistry structures and properties of 134 kilo molecules. Scientific data, 1(1):1-7, 2014.

The Network

Training has been done over 300 epochs with a batch size of 128. Loss function: Mean-Square Error (MSE).

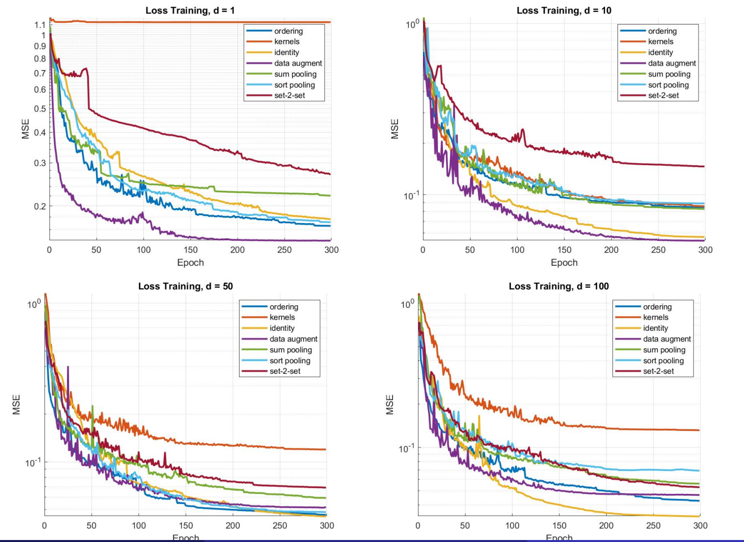
The same 7 α modules have been tested:

1. identity: $\alpha(X) = X$; no permutation invariance.
2. data augmentation: $\alpha(X) = X$ BUT the training data set has been augmented with 4 random permutatons of each graph.
3. ordering: $\alpha(X) = \downarrow(XA)$, $A = [I\ 1]$
4. kernels: $\alpha(X) = (\sum_{k=1}^{n} exp(-\|x_k - a_j\|^2))_{1 \leq j \leq m=5nd}$
5. sumpooling: $\alpha(X) = 1^T X$
6. sort-pooling: sorted by last column
7. set-to-set: introduced in [Vinyals&al.][28]

[28] Vinyals, O., Bengio, S. Kudlur, M., Order Matters: Sequence to sequence for sets, ICLR 2016

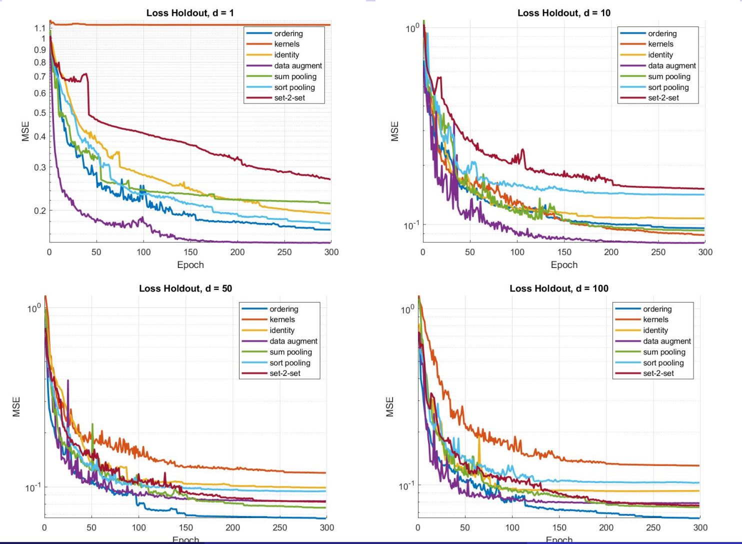
QM9 Regression Example

Training MSE



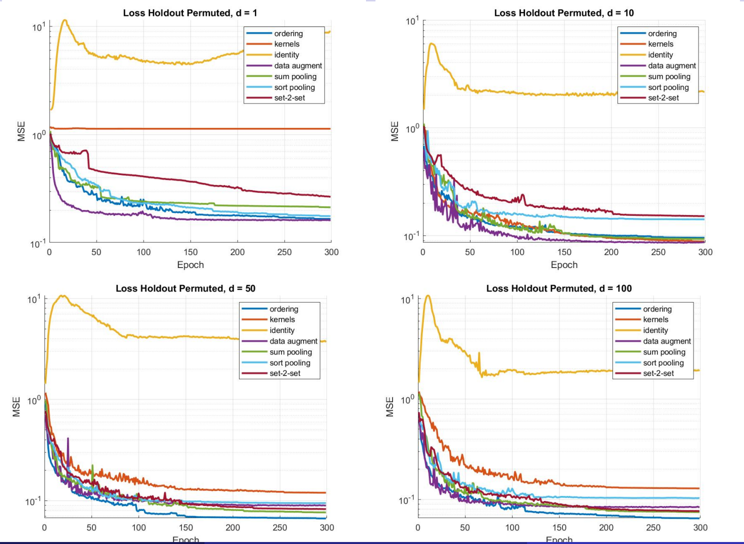
QM9 Regression Example

Validation MSE



QM9 Regression Example

Validation MSE with Random Permutations



Performance Results: MAE

d = 100	ordering	kernels	identity	data augment	sum-pooling	sort-pooling	set-2-set
Training	0.155	0.269	0.139	0.164	0.178	0.199	0.173
Holdout	0.187	0.267	0.227	0.206	0.201	0.239	0.201
Holdout Perm	0.187	0.267	1.086	0.213	0.201	0.239	0.201

Table: Mean Absolute Error (MAE) for regression of the electron energy gap $\Delta\varepsilon = LUMO - HOMO$ (eV) of the seven algorithms on QM9 dataset after 300 epochs for embedding dimension $d = 100$

For comparison:

- chemical accuracy is $0.043eV$
- the best ML method [Gilmer&al.] achieves MAE of $0.053eV$
- Coulomb method [Rupp&al.] achieves MAE of $0.229eV$

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A Universal Embedding

Consider the map

$$\mu : \widehat{\mathbb{R}^{n\times d}} \to \mathcal{P}(\mathbb{R}^d) \ \ , \ \ \mu(X)(x) = \frac{1}{n}\sum_{k=1}^{n} \delta(x - x_k)$$

where $\mathcal{P}(\mathbb{R}^d)$ denotes the convex set of probability measures over $\mathbb{R}^d$, and δ denotes the Dirac measure. x_k is the k^{th} row of X.
Clearly $\mu(X') = \mu(X)$ iff $X' = PX$ for some $P \in \mathcal{S}_n$.
The Wasserstein-2 distance is equivalent to the natural metric:

$$W_2(\mu(X),\mu(Y))^2 := \inf_{q\in J(\mu(X),\mu(Y))} \mathbb{E}_q[\|x-y\|_2^2] = \min_{P\in\mathcal{S}_n} \|Y - PX\|^2$$

By Kantorovich-Rubinstein theorem, the Wasserstein-1 distance (the Earth moving distance) extends to a norm on the space of signed Borel measures.
Main drawback: $\mathcal{P}(\mathbb{R}^d)$ is infinite dimensional!

Finite Dimensional Embeddings

Idea: “Project” the measure onto a finite dimensional space. This is accomplished by *kernel methods*:

Fix a family of functions $f_1, \cdots, f_m$ and consider:

$$\mu(X) \mapsto \int_{\mathbb{R}^d} f_j(x) d\mu(X) = \frac{1}{n} \sum_{k=1}^{n} f_j(x_k) \quad , \quad j \in [m]$$

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Possible choices:

1. Polynomial embeddings: $\mathbb{R}[X]^{\mathcal{S}_n}$, ring of invariant polynomials; [Lipman&al.],[Peyré&al.],[Sanay&al.],[Kemper book] ...
2. Gaussian kernels: $f_j(x) = exp(-\|x - a_j\|^2/\sigma_j^2)$; [Gilmer&al.],[Zaheer&al.], [Vinyals&al.],...
3. Fourier kernels (cmplx embd): $f_j(x) = exp(2\pi i \langle x, \omega_j \rangle)$; related to Prony method; [Li&Liao] for bi-Lipschitz estimates.

Main drawback: No global bi-Lipschitz embeddings [Cahill&al.]. Ok on (some) compacts.

The Embedding Problem

Notations (2)

Definition

Fix $X \in \mathbb{R}^{n\times d}$. A matrix $A \in \mathbb{R}^{d\times D}$ is called admissible for X if $\beta_A^{-1}(\beta_A(X)) = \hat{X}$. In other words, if $Y \in \mathbb{R}^{n\times d}$ so that $\downarrow(XA) = \downarrow(YA)$ then there is $\Pi \in \mathcal{S}_n$ sot that $Y = \Pi X$.

We denote by $\mathcal{A}_{d,D}(X)$ (or $\mathcal{A}(X)$) the set of admissible keys for X.

Definition

Fix $A \in \mathbb{R}^{d\times D}$. A data matrix $X \in \mathbb{R}^{n\times d}$ is said separated by A if $A \in \mathcal{A}(X)$.

We let $\mathcal{S}(A)$ denote the set of data matrices separated by A.
The key A is universal iff $\mathcal{S}(A) = \mathbb{R}^{n\times d}$.

Genericity Results for $d \geq 2$

Admissible keys

Theorem

Let $X \in \mathbb{R}^{n\times d}$. For any $D \geq d+1$ the set $\mathcal{A}_{d,D}(X)$ of admissible keys for X is dense in $\mathbb{R}^{d\times D}$ with respect to Euclidean topology, and it is generic with respect to Zariski topology. In particular, $\mathbb{R}^{d\times D} \setminus \mathcal{A}_{d,D}(X)$ has Lebesgue measure 0, i.e., almost every key is admissible for X.

Proof

It is sufficient to consider the case $D = d+1$. Also, it is sufficient to analyze the case $A = [I_d \;\; b]$ and to show that a generic $b \in \mathbb{R}^d$ defines an admissible key. The vector $b \in \mathbb{R}^d$ does **not** define an admissible key if there are $\Xi, \Pi_1, \cdots, \Pi_d \in S_n$ so that for $Y = [\Pi_1 x_1, \cdots, \Pi_d x_d]$,

$$Yb = \Xi X b \;\text{ but }\; Y - \Pi X \neq 0 \;,\; \forall \Pi \in \mathcal{S}_n$$

Define the linear operator

Genericity Results for $d \geq 2$

Admissible keys

Proof - cont'd

Let

$$\mathcal{P} = \left\{(\Pi_1, \cdots, \Pi_d) \in (\mathcal{S}_n)^d \;\; \forall \Pi \in \mathcal{S}_n, \exists k \in [d] \; s.t. \; (\Pi - \Pi_k)x_k \neq 0\right\}$$

Then

$$\{b \in \mathbb{R}^d : [I_d \; b] \text{ not admissible for } X\} = \bigcup_{(\Xi;\Pi_1,\cdots,\Pi_d)\in\mathcal{S}_n\times\mathcal{P}} \ker(B(\Xi;\Pi_1,\cdots,\Pi$$

It is now sufficient to show that each null space has dimension less than d. Indeed, the alternative would mean $B(\Xi;\Pi_1,\cdots,\Pi_d) = 0$ but this would imply $(\Pi_1,\cdots,\Pi_d) \notin \mathcal{P}$. $\square$

Non-Universality of vector keys

Insufficiency of a single vector key

The following is a no-go result, which shows that there is no universal single vector key for data matrices tall enough.

Proposition

If $d \geq 2$ and $n \geq 3$,

$$\bigcup_{X \in \mathbb{R}^{n \times d}} \{b \in \mathbb{R}^d : \ A = [I_d \ \ b] \text{ not admissible for} X\} = \mathbb{R}^d.$$

Consequently,

$$\bigcap_{X \in \mathbb{R}^{n \times d}} \mathcal{A}_{d,d+1}(X) = \emptyset.$$

On the other hand, for $n = 2$, $d = 2$, any vector $b \in \mathbb{R}^2$ with $b_1 b_2 \neq 0$ defines a universal key $A = [I_2 \ \ b]$.

Non-Universality of vector keys

Insufficiency of a single vector key - cont'd

Proof

To show the result, it is sufficient to consider a counterexample for $n = 3$, $d = 2$, with key $b = [1, 1]^T$.

$$X = \begin{bmatrix} 1 & -1 \\ -1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad Y = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix}$$

Then $Xb = [0, -1, 1]^T$ and $Yb = [1, 0, -1]^T$, yet $X \not\sim Y$. Thus $[I_2\ b]$ is not admissible for X.

Then note if $a \in \mathbb{R}^d$ so that $[I_d\ a]$ is admissible for X then for any $P \in S_d$ and L an invertible $d \times d$ diagonal matrix, $L^{-1}P^TA \in \mathcal{A}_{d,1}(XPL)$. This shows how for any $b \in \mathbb{R}^2$, one can construct $X \in \mathbb{R}^{3\times 2}$ so that $b \notin \mathcal{A}_{2,1}(X)$.

For $n > 3$ or $d > 2$, proof follows by embedding this example.

Genericity Results for $d \geq 2$

Admissible Data Matrices

Theorem

Assume $a \in \mathbb{R}^d$ is a vector with non-vanishing entries, i.e., $a_1 a_2 \cdots a_d \neq 0$. Then for any $n \geq 1$, $\mathcal{S}([I_d \ \ a])$ is dense in $\mathbb{R}^{n \times d}$ and includes an open dense set with respect to Zariski topology. In particular, $\mathbb{R}^{n \times d} \setminus \mathcal{S}([I_d \ \ a])$ has Lebesgue measure 0, i.e., almost every data matrix X is separated by the vector key a.

Genericity Results for $d \geq 2$

Admissible Data Matrices

Theorem

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Corollary

Assume $A \in \mathbb{R}^{d \times (D-d)}$ is a matrix such that at least one column has non-vanishing entries. Then for any $n \geq 1$, $\mathcal{S}([I_d \;\; A])$ is dense in $\mathbb{R}^{n \times d}$ and is generic with respect to Zariski topology. In particular, $\mathbb{R}^{n \times d} \setminus \mathcal{S}([I_d \;\; A])$ has Lebesgue measure 0, i.e., almost every data matrix X is separated by the matrix key $[I_d \;\; A]$.

Proof that $\mathcal{S}([I_d\ A])$ is generic

The case $D > d$

Assume $A \in \mathbb{R}^{d\times(D-d)}$ satisfies $A_{1,k}A_{2,k}\cdots A_{d,k} \neq 0$ for some $k \in [D-d]$. The set of non-separated data matrices $X \in \mathbb{R}^{n\times d}$ (i.e., the complement of $\mathcal{S}([I_d\ A])$) factors as follows:

$$\mathbb{R}^{n\times d}\backslash\mathcal{S}([I_d\ A]) = \bigcup_{(\Xi_1,\cdots,\Xi_{D-d};\Pi_1,\cdots,\Pi_d)\in(\mathcal{S}_n)^D} \left(ker\ L(\Xi_1,\cdots,\Xi_{D-d};\Pi_1,\cdots,\Pi_d;A\right.$$

$$\left.\setminus \bigcup_{\Pi\in\mathcal{S}_n} \ker\ M(\Pi,\Pi_1,\cdots,\Pi_d)\right) \quad (*)$$

where, with $A = [a_1,\cdots,a_{D-d}]$, $X = [x_1,\cdots,x_d]$:

$$L(\Xi_1,\cdots,\Xi_{D-d};\Pi_1,\cdots,\Pi_d;A):\mathbb{R}^{n\times d}\rightarrow\mathbb{R}^{n\times D-d}\ ,\ (L((\ldots)X)_k=[(\Xi_k-\Pi_1)x_1,\cdots,(\Xi_k-\Pi_d)x_d]a_k\ ,\ k\in[D-$$

$$M(\Pi,\Pi_1,\cdots,\Pi_d):\mathbb{R}^{n\times d}\rightarrow\mathbb{R}^{n\times d}\ ,\ M(\Pi,\Pi_1,\cdots,\Pi_d)X=[(\Pi-\Pi_1)x_1,\cdots,(\Pi-\Pi_d)x_d]$$

Proof that $\mathcal{S}(A)$ is generic

cont'd

1. The outer union can be reduced by noting that on the "diagonal" Δ,

$$\Delta = \{(\Xi_1, \cdots, \Xi_{D-d}; \Pi_1, \cdots, \Pi_d) \in (\mathcal{S}_n)^D \quad , \quad \Pi_1 = \Pi_2 = \cdots = \Pi_d\}$$

$$M(\Pi_1, \Pi_1, \cdots, \Pi_d) = 0 \rightarrow \bigcup_{\Pi \in \mathcal{S}_n} \ker M(\Pi, \Pi_1, \cdots, \Pi_d) = \mathbb{R}^{n \times d}$$

2. If $(\Xi_1, \cdots, \Xi_{D-d}; \Pi_1, \cdots, \Pi_d) \in (\mathcal{S}_n)^D \setminus \Delta$ then for every $k \in [D-d]$ there is $j \in [d]$ such that $\Xi_k - \Pi_j \neq 0$. In particular choose the k column of A that is non-vanishing. Let $x_j \in \mathbb{R}^n$ so that $(\Xi_k - \Pi_j)x_j \neq 0$. Consider the matrix $X = [0, \cdots, 0, x_j, 0, \cdots, 0]$ where x_j is the only non identically 0 column. Claim: $X \notin \ker L(\Xi_1, ..., \Pi_d; A)$. Indeed, the resulting k column of $L()X$ is $A_{j,k}(\Xi_k - \Pi_j)x_j \neq 0$. It follows that

$$\dim \ker L(\Xi_1, \cdots, \Xi_{D-d}; \Pi_1, \cdots, \Pi_d; A) < nd$$

Hence $\mathbb{R}^{n \times d} \setminus \mathcal{S}([I_d \; A])$ is a finite union of subsets of closed linear spaces properly included in $\mathbb{R}^{n \times d}$. This proves the theorem. $\square$

Additional Relations

Note the following relationship and matrix representation of X when matrices are column-stacked:

$$M(\Pi, \Pi_1, \cdots, \Pi_d) = L(\Pi, \cdots, \Pi; \Pi_1, \cdots, \Pi_d; I)$$

$$L \equiv \begin{bmatrix} A_{1,1}(\Xi_1 - \Pi_1) & A_{2,1}(\Xi_1 - \Pi_2) & \cdots & A_{d,1}(\Xi_1 - \Pi_d) \\ A_{1,2}(\Xi_2 - \Pi_1) & A_{2,2}(\Xi_2 - \Pi_2) & \cdots & A_{d,2}(\Xi_2 - \Pi_d) \\ \vdots & \vdots & \ddots & \vdots \\ A_{1,D-d}(\Xi_{D-d} - \Pi_1) & A_{2,D-d}(\Xi_{D-d} - \Pi_2) & \cdots & A_{d,D-d}(\Xi_{D-d} - \Pi_d \end{bmatrix}$$

a $n(D-d) \times nd$ matrix.

Towards universal keys

The arXiv preprint provides necessary and sufficient conditions for a key to be universal.

Open Problem: Given (n, d) find the smallest dimension D so that there exists a universal key $A \in \mathbb{R}^{d \times D}$ for $\mathbb{R}^{n \times d}$.

So far we obtained (joint with Daniel Levy (UMD)):

n	d	D-d
2	2	1
3	2	2
4	2	2
5	2	3
6	2	≥ 4

Open Problem: If a universal key exists for a triple (n, d, D) then is it true that universal keys are generic in $\mathbb{R}^{d \times D}$?

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