G-Invariant Representations and their Lipschitz Properties

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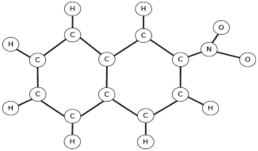
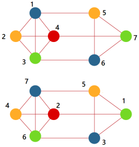
Motivation

Certain phenomena and systems enjoy invariance to group actions.

In physics: the celebrated Noether theorem asserts that a conservation law exists for any symmetry (i.e., group invariance) of the Hamiltonian.

In data science, certain systems exhibit intrinsic invariance to group actions: in graph deep learning, graph level regression and classification must be invariant to node labeling. Specifically, this means: if (W, X) is a data graph, where $W \in Sym(\mathbb{R}^n)$ and $X \in \mathbb{R}^{n \times d}$, then for any $n \times n$ permutation matrix P, the regression/classification function f, $(W, X) \mapsto f(W, X)$ must satisfy $f(PWP^T, PX) = f(W, X)$.





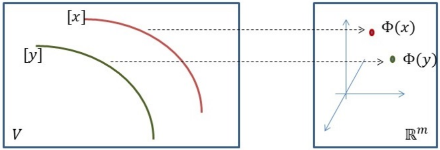
Problem Formulation

Consider a group $G \subset O(d)$ acting on the Euclidean space $V = \mathbb{R}^d$.

General problem

Construct an embedding map $\Phi : V \to \mathbb{R}^m$

1. Invariance: $\Phi(U_g x) = \phi(x)\ \forall g \in G, x \in V$
2. Injectivity: if $\Phi(x) = \Phi(y)$ then there exists $g \in G$ so that $y = U_g x$.
3. Φ is bi-Lipschitz on $(\hat{V} = V/G, \mathbf{d})$, where $\mathbf{d}([x],[y]) = \inf_{u \in [x], v \in [y]} \|u - v\|$.



Approaches

Over the past years, several constructions have been proposed:

1. Invariant Polynomials: Hilbert, Noether, ..., Cahill[1], Bandeira[2]
2. Kernels: replace monomials by other kernels, e.g. $e^{i\omega x}$, e^{-x^2}, $\sigma(\langle x, a\rangle)$[3]
3. Sorting: extends the 1-D sorting, $x \mapsto \downarrow x$ [4,5]

1+2: *sum pooling* layer; 3: extension of *max pooling* layer in deep nets[6], [7].

[1] J. Cahill, A. Contreras, A.C. Hip, Complete Set of translation Invariant Measurements with Lipschitz Bounds, Appl. Comput. Harm. Anal. 49 (2020), 521–539.

[2] A. Bandeira, B. Blum-Smith, J. Kileel, J. Niles-Weed, A. Perry, A.S. Wein, Estimation under group actions: Recovering orbits from invariants, ACHA 66 (2023)

[3] D. Yarotsky, Universal approximations of invariant maps by neural networks, Constructive Approximation (2021)

[4] R. Balan, N. Haghani, M.Singh, Permutation Invariant Representations with Applications to Graph Deep Learning, arXiv:2203.07546

[5] J. Cahill, J.W. Iverson, D.G. Mixon, D. Packer, Group-invariant max filtering, arXiv:2205.14039.

[6] O. Vinyals, S. Bengio, M. Kudlur, Order Matters: Sequence to sequence for sets, Proc. ICLR 2016

Existing Results

Injectivity problem

Over the past 15 years or so, there have been works that recognized the difference between *generating polynomials* and *separating invariants*[8]
A seminal paper that resurfaces results on semi-algebraic sets is [9]. The method goes back to earlier works in phase retrieval[10].
More recently, in the context of G-invariance, [11],[12], or permutation invariance[13]

[8]Emilie Dufresne, Separating invariants and finite reflection groups, Advances in Mathematics 221 (2009), no. 6, 1979–1989.

[9]Dym Nadav, Steven J. Gortler. "Low dimensional invariant embeddings for universal geometric learning." arXiv preprint arXiv:2205.02956.

[10]R. Balan, P. Casazza, D. Edidin, On signal reconstruction without phase, ACHA 20(2006)

[11]D. G. Mixon, D. Packer, Max filtering with reflection groups, arXiv:2212.05104

[12]R. Balan, E. Tsoukanis, G-invariant representations using coorbits: Injectivity properties, arXiv:2310.16365

[13]On the equivalence between graph isomorphism testing and function approximation with GNNs, Z. Chen, S. Villar, L. Chen, J. Bruna, NeurIPS 2019.

Existing Results (2)

Lipschitz and Bi-Lipschitz properties

Earlier results obtain Lipschitz/bi-Lipschitz properties on compacts, or certain classes of functions.

Global L/bi-L are harder to establish and typically rule out polynomial based embeddings.

So far only sorting based embeddings showed such global properties [14],[15], [16]

[14] R. Balan, E. Tsoukanis, G-invariant representations using coorbits: Bi-lipschitz properties, arXiv:2308.11784

[15] J. Cahill, J. W. Iverson, D. G. Mixon, Bilipschitz group invariants, arXiv:2305.17241

[16] D. G. Mixon, Y. Qaddura, Injectivity, stability, and positive definiteness of max filtering, arXiv:2212.11156

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Coorbit Representations

Let V be a d-dimensional Hilbert space and G a finite group of size $N = |G|$ acting unitarily on V, $\{U_g, g \in G\}$.
The quotient space $\widehat{V} = V/G$ is the set of orbits $[x] = \{U_g x \ , \ g \in G\}$ induced by the group action, where for $x, y \in V$, $x \sim y$ iff $y = U_g x$ for some $g \in G$. $(\widehat{V}, \mathbf{d})$ becomes a metric space with the natural distance

$$\mathbf{d}([x],[y]) = \min_{g \in G} \|x - U_g y\|$$

Fix a generator $w \in V$ (call it, window or template) and consider the nonlinear map induced by sorting its coorbit:

$$\phi_w : V \to \mathbb{R}^N \quad , \quad \phi_w(x) = \downarrow \left((\langle x, U_g w \rangle)_{g \in G} \right).$$

where $\downarrow(y) = (y_{\pi(i)})_{i \in [N]}$ is the non-increasing sorting operator: $y_{\pi(1)} \geq \cdots \geq y_{\pi(N)}$.

Invariant Coorbit Representations

For a collection $\mathbf{w} = (w_1, \ldots, w_p) \in V^p$ let

$$\Phi_{\mathbf{w}} : V \to \mathbb{R}^{N \times p} \quad , \quad \Phi_{\mathbf{w}}(x) = \left[\phi_{w_1}(x) | \cdots | \phi_{w_p}(x)\right] .$$

For a subset $S \subset [N] \times [p]$ of cardinal $m = |S|$, let

$$\Phi_{\mathbf{w},S} : V \to l^2(S) \sim \mathbb{R}^m \ , \ \Phi_{\mathbf{w},S}(x) = (\Phi_{\mathbf{w}}(x))|_S$$

be the restriction of $\Phi_{\mathbf{w}}$ to S. For a linear operator $\mathcal{L} : l^2(S) \to \mathbb{R}^m$, let

$$\Psi_{\mathbf{w},S,\mathcal{L}} : V \to \mathbb{R}^m \quad , \quad \Psi_{\mathbf{w},\mathcal{L}}(x) = \mathcal{L}(\Phi_{\mathbf{w},S}(x))$$

be the "projection" of $\Phi_{\mathbf{w},S}$ through $\mathcal{L}$ into $\mathbb{R}^m$.
Problems: Construct $(\mathbf{w}, S)$ so that $\Phi_{\mathbf{w},S}$ is a bi-Lipschitz embedding of $\widehat{V}$. Construct $(\mathbf{w}, S, \mathcal{L})$ so that $\Psi_{\mathbf{w},S,\mathcal{L}}$ is bi-Lipschitz.

Invariant Coorbit Representations (2)

Special cases:
1. If $G = S_n$ and $V = \mathbb{R}^{n\times d}$ with action $(P, X) \mapsto PX$, then [17] introduced the embedding $\beta_A(X) = \downarrow(XA)$, for *key* $A \in \mathbb{R}^{d\times D}$ and sorting operator acting independently in each column.
Equivalent recasting: Let $w_1 = \delta_1 \cdot a_1^T, \ldots, w_D = \delta_1 \cdot a_D^T$, where $\delta_1 = (1, 0, \ldots, 0)^T$ and $A = [a_1 | \cdots | a_D]$. Then note $\phi_{w_1}(X) = \downarrow(Xa_1) \otimes 1_{(n-1)!}$. Thus $\Phi_{\mathbf{w}}(X) = \beta_A(X) \otimes 1_{(n-1)!}$. Thus $\beta_A(X) = \Phi_{\mathbf{w},S}(X)$ for an appropriate subset $S \subset [n!] \times [D]$ of size nD.
2. The *max filter* introduced in [18] for some template $w \in V$ is defined by $\langle\langle \cdot, w \rangle\rangle : V \to \mathbb{R}$, $\langle\langle x, w \rangle\rangle = \max_{g\in G} \langle x, U_g w \rangle$. Equivalent recasting: $\langle\langle x, w \rangle\rangle = \Phi_{w,S}(X)$, for $S = \{1\}$.

[17] R. Balan, N. Haghani, M.Singh, Permutation Invariant Representations with Applications to Graph Deep Learning, arXiv:2203.07546 (2022)

[18] J. Cahill, J. W. Iverson, D. G. Mixon, D. Packer, Group-invariant max filtering, arXiv:2205.14039 (2022)

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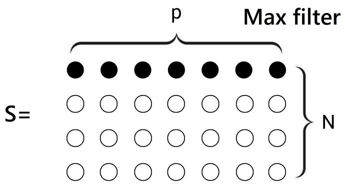
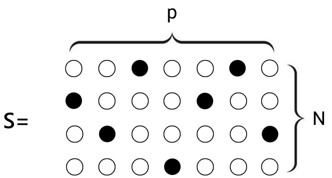
Minimal embeddings

Setup: Let V be a d-dimensional Hilbert space and G a finite group of size $N=|G|$ acting unitarily on V, $\{U_g, g\in G\}$. For a subset $S\subset[N]\times[p]$ of cardinal $m=|S|$, let

$$\Phi_{\mathbf{w},S}:V\to l^2(S)\sim\mathbb{R}^m \ , \ \ \Phi_{\mathbf{w},S}(x)=(\Phi_{\mathbf{w}}(x))|_S$$

be the restriction of $\Phi_{\mathbf{w}}$ to S.





A typical injectivity result asserts that for $p\geq p_{min}$ and a *generic* $\mathbf{w}\in V^p$, for any S of cardinal $m\geq m_{min}$ that satisfy certain shape conditions, the map $\Phi_{\mathbf{w},S}$ is injective on $\hat{V}$. (p_{min},m_{min}) depend on specific rep.

Current injectivity results

Setup: $dim(V) = d$, G finite group of size N. Denote by $d_G = \dim\{x \in V \; ; \; U_g x = x \; \forall g \in G\}$.

Theorem (R.B, E.Tsoukanis '23)

Let $p \geq 2d - d_G$. Then for a generic $\mathbf{w} \in V^p$ the following holds: for every $S \subset [N] \times [p]$ that satisfies the property that for every $i \in [p]$ there is a $k \in [N]$ so that $(k, i) \in S$, the map $\Phi_{\mathbf{w},S}$ is injective on $\hat{V}$.

Theorem (R.B, E.Tsoukanis '23)

If $S \subset [N] \times [p]$ satisfies a stronger intersection property, namely, for each $i \in [p]$ there are distinct $k_1, \ldots, k_n \in [N]$ so that $(k_1, i), \ldots, (k_n, i) \in S$, then the lower bound $2d - d_G$ can be replaced by

$$p_{n,min} = \max_{q \in [n], g,h \in G^n} \frac{1}{q}(\gamma_q^{g,h} - d_G - 1) \text{ where}$$

$$\gamma_q^{g,h} = semialg{-}dim\{(x, y) \in V \times V : dim(span\{\, U_{g(1)}x - U_{h(1)}y, \ldots, U_{g(n)}x - U_{h(n)}y\}) \leq q\}.$$

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Injectivity implies (bi-Lipschitz) Stability

Theorem

For fixed $\mathbf{w} \in V^p$ *and* $S \subset [N] \times [p]$, *where* $|S| = m$, *suppose that the map* $\Phi_{\mathbf{w},S} : V \to \mathbb{R}^m$, *is injective on* V/G. *Then,* $\exists 0 < a \leq b < \infty$ *such that* $\forall (x, y) \in V,\ x \nsim y$

$$a\,\mathrm{d}([x],[y]) \leq \|\Phi_{\mathbf{w},S}(x) - \Phi_{\mathbf{w},S}(y)\|_2 \leq b\,\mathrm{d}([x],[y]).$$

Injectivity implies (bi-Lipschitz) Stability

Theorem

For fixed $\mathbf{w} \in V^p$ *and* $S \subset [N] \times [p]$*, where* $|S| = m$*, suppose that the map* $\Phi_{\mathbf{w},S} : V \to \mathbb{R}^m$*, is injective on* V/G*. Then,* $\exists 0 < a \leq b < \infty$ *such that* $\forall (x, y) \in V, \; x \nsim y$

$$a\,\mathrm{d}([x],[y]) \leq \|\Phi_{\mathbf{w},S}(x) - \Phi_{\mathbf{w},S}(y)\|_2 \leq b\,\mathrm{d}([x],[y]).$$

Corollary

For max filter bank $\Phi : \mathbb{R}^d/G \to \mathbb{R}^m$*, injectivity implies stability.*

Upper Lipschitz bound

Lemma

Let $\mathbf{w} \in V^p$, $S \subset [N] \times [p]$ *and*

$$B = \max_{\substack{\sigma_1,\ldots,\sigma_p \subset G \\ |\sigma_i| = m_i, \forall i}} \lambda_{max} \left(\sum_{i=1}^{p} \sum_{g \in \sigma_i} g.w_i w_i^T U_g^T \right)$$

where $S_i = \{j \in [N], (i,j) \in S\}$ *and* $m_i = |S_i|$. *Then* $\Phi_{\mathbf{w},S} : \hat{V} \to \mathbb{R}^m$ *is Lipschitz with constant upper bounded by* $\sqrt{B}$.

Lower Lipschitz bound

The proof of the main Theorem is by contradiction.
1. If lower Lipschitz constant vanishes, then it must vanish locally: there are $(x_n)_n, (y_n)_n$ such that

$$\lim_{n\to\infty} \frac{\|\Phi_{\mathbf{w},S}(x_n) - \Phi_{\mathbf{w},S}(y_n)\|^2}{\mathbf{d}([x_n],[y_n])^2} = 0$$

and

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n = z_1,\ \ \|x_n\| = 1,\ \ \|y_n\| \leq 1,\ \ \|z_1\| = 1$$

and they are aligned with one another:

$$\|x_n - y_n\| = \min_{g\in G} \|x_n - U_g y_n\| \tag{1}$$

$$\|x_n - z_1\| = \min_{g\in G} \|x_n - U_g z_1\| \tag{2}$$

$$\|y_n - z_1\| = \min_{g\in G} \|y_n - U_g z_1\| \tag{3}$$

Lower Lipschitz bound

2. We construct inductively $z_2, z_3, ..., z_d$ such that for all $1 \leq k \leq d-1$:

$$\|z_{k+1}\| \ll \|z_k\|, \;\; \dim(\mathrm{span}(z_1, \ldots, z_k)) = k$$

and the local lower Lipschitz constant vanishes in a convex set $\{\sum_{r=1}^{k} a_r z_r \;,\; |a_r - 1| < \epsilon\}$.
3. For $k = d$ this construction defines a non-empty open set $\{\sum_{r=1}^{k} a_r z_r \;,\; |a_r - 1| < \epsilon\}$ where the local lower Lipschitz constant vanishes.
4. Finally, we can construct $u, v \neq 0$, so that $x = u + \sum_{r=1}^{d} z_r$ and $y = v + \sum_{r=1}^{d} z_r$ satisfy $x \neq y$ and yet

$$\Phi_{\mathbf{w},S}(x) = \Phi_{\mathbf{w},S}(y).$$

This contradicts the injectivity hypothesis.

Happy Birthday Charly!

Thank you!