p.48/#26: $y' = 2(1+x)(1+y^2)$, $y(0)=0$

$$\frac{dy}{1+y^2} = 2(1+x)\,dx$$

$$\arctan y = (1+x)^2 + C$$

$$0 = \arctan 0 = (1+0)^2 + C = 1 + C \Rightarrow C = -1$$

$$y = \tan\left[(1+x)^2 - 1\right] = \tan(x^2+2x)$$

min value occurs where x^2+x takes min value, i.e, where $x=-1$.

p.451/#1a (just for $t=0.1$): 2 steps of Euler

$$y_{n+1} = y_n + h f(t_n, y_n), \quad f(t,y) = 3+t-y, \quad y(0)=1, \quad h=0.05$$

$t_0=0$, $y_0=1$, $y_1 = y_0 + (0.05)(3+0-1) = 1+0.05(2) = 1.1$

$$y_2 = y_1 + h f(t_1, y_1) = 1.1 + 0.05(3+0.05-1.1)$$

$$= 1.1 + 0.05(1.95) = 1.1 + 0.0975 = 1.1975$$

p.163/#22: $y'' + 2y' + 2y = 0$, $y(\frac{\pi}{4}) = 2$, $y'(\frac{\pi}{4}) = -2$

$$r^2+2r+2 = (r+1)^2+1, \quad r = -1 \pm i$$

G.S. $y = e^{-t}(c_1 \cos t + c_2 \sin t)$

$$y' = e^{-t}(-c_1 \sin t + c_2 \cos t - c_1 \cos t - c_2 \sin t)$$

$$2 = y(\tfrac{\pi}{4}) = e^{-\pi/4}(c_1+c_2)\tfrac{1}{2}\sqrt{2}$$

$$-2 = y'(\tfrac{\pi}{4}) = e^{-\pi/4}(-2c_1)\tfrac{1}{2}\sqrt{2}$$

$$\Rightarrow c_1 = \sqrt{2}\,e^{\pi/4}$$

$$\Rightarrow 2 = e^{-\frac{\pi}{4}}\tfrac{1}{2}\sqrt{2}\left(e^{\frac{\pi}{4}}\sqrt{2} + c_2\right)$$

$$2 = 1 + e^{-\frac{\pi}{4}}\tfrac{1}{2}\sqrt{2}\,c_2$$

$$c_2 = e^{\pi/4}\sqrt{2}$$

$$y = \sqrt{2}\,e^{-(t-\pi/4)}(\cos t + \sin t)$$

a decreasing oscillation as t increases p.155/#12: $(x-2)y'' + y' + (x-2)(\tan x)y = 0$, $y(3)=1$, $y'(3)=2$

There is a unique soln valid in the largest interval containing $t=3$ on which the coeff fctns are continuous, but to apply that result, the ODE must first be normalized.

$$y'' + \frac{1}{x-2}y' + (\tan x)y = 0$$

The coeffs blow up at $x=2$ and at the singular points of $\tan x$, ie, $x = n\frac{\pi}{2}$, n odd integer.

The closest singularity to the left of $t=3$ is $t=2$ since $\frac{\pi}{2}<2$ and the closest to the right is $\frac{3\pi}{2}$. So the interval of definition is $2<t<\frac{3\pi}{2}$.

p.183/#15: $y'' - 2y' + y = te^t + 4$, $y(0)=1$, $y'(0)=1$

First: Gen Soln of homog: $r^2-2r+1=(r-1)^2$ $\quad y=c_1e^t+c_2te^t$

Second: particular soln of inhomog; undet coeffs candidate $e^t(at+b)+c$

but 1st two are solns of homog, so must use $t^2e^t(at+b)+c$

$$y = e^t(at^3+bt^2)+c$$

$$y' = e^t(at^3+bt^2+3at^2+2bt) = e^t(at^3+(b+3a)t^2+2bt)$$

$$y'' = e^t(at^3+(b+3a)t^2+2bt + 3at^2+2(b+3a)t+2b)$$

$$= e^t(at^3+(b+6a)t^2+(4b+6a)t+2b)$$

$$y''-2y'+y = e^t(6at+2b)+c = te^t+4$$

$$\Rightarrow a=\tfrac{1}{6},\ b=0,\ c=4$$

GS of inhomog $y=c_1e^t+c_2te^t+\frac{1}{6}t^3e^t+4$

$y(0)=1=c_1+4$, $c_1=-3$

$y' = -3e^t + c_2e^t(t+1) + \frac{1}{6}e^t(t^3+3t^2)$

$y'(0)=1=-3+c_2$, $c_2=4$

Ans $y=-3e^t+4te^t+\frac{1}{6}t^3e^t+4$.

p. 59 / Sect. 2.3 / # 2

Incoming:

γ g/L of salt

2 L/min

Outgoing: 2 L/min of mix

Volume: 120 L

Let $s(t)$ denote the salt in tank, at time t, in grams.

$$\begin{cases} \dfrac{ds}{dt} = 2\gamma - 2\dfrac{s}{120} \\ s(0) = 0 \end{cases}$$

Solve by integrating factor: $\dfrac{ds}{dt} + \dfrac{1}{60} s = 2\gamma$

$$e^{\frac{t}{60}} \cdot \frac{ds}{dt} + \frac{1}{60} e^{\frac{t}{60}} s = e^{\frac{t}{60}} \cdot 2\gamma$$

$$\frac{d}{dt}\left(e^{\frac{t}{60}} \cdot s(t)\right) = 2\gamma e^{t/60}$$

$$e^{\frac{t}{60}} \cdot s(t) - 0 = \int_0^t 2\gamma e^{t/60}\, dt = 120\gamma\left(e^{\frac{t}{60}} - 1\right)$$

$$\Rightarrow \quad \underline{s(t) = 120 \cdot \gamma\left(1 - e^{-\frac{t}{60}}\right)}$$

$$\lim_{t\to\infty} s(t) = 120 \cdot \gamma$$

Section 6.3
p. 323 / #24

$$F(s) = \frac{e^{-s} + e^{-2s} - e^{-3s} - e^{-4s}}{s}$$

$$f(t) = \mathcal{L}^{-1}\{F\} = u_1(t) + u_2(t) - u_3(t) - u_4(t) = \begin{cases} 0, & 0 \le t < 1 \\ 1, & 1 \le t < 2 \\ 2, & 2 \le t < 3 \\ 1, & 3 \le t < 4 \\ 0, & 4 \le t \end{cases}$$

(we used: $\mathcal{L}^{-1}\{\frac{e^{-cs}}{s}\} = u_c$).

p. 389 / Section 7.4 / #6

$$\underline{x}^{(1)}(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}, \quad \underline{x}^{(2)}(t) = \begin{pmatrix} t^2 \\ 2t \end{pmatrix}$$

(a) $W[\underline{x}^{(1)}, \underline{x}^{(2)}](t) = \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = 2t^2 - t^2 = t^2$

(b) Intervals where $\underline{x}^{(1)}(t)$ and $\underline{x}^{(2)}(t)$ are independent: $(-\infty, 0)$, $(0, \infty)$.

[$\underline{x}^{(1)}(0)$ and $\underline{x}^{(2)}(0)$ are not independent: $0 \cdot \underline{x}^{(1)}(0) + 1 \cdot \underline{x}^{(1)}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$]

(c) Coefficients of the homogeneous system $\underline{x}' = \underline{P}(t)\,\underline{x}$ will have a singularity at $t = 0$.

(d) Compute $P(t)$:

$$\begin{bmatrix} 1 & 2t \\ 0 & 2 \end{bmatrix} = P(t) \cdot \begin{bmatrix} t & t^2 \\ 1 & 2t \end{bmatrix}$$

$$P(t) = \begin{bmatrix} 1 & 2t \\ 0 & 2 \end{bmatrix} \cdot \begin{bmatrix} t & t^2 \\ 1 & 2t \end{bmatrix}^{-1} = \frac{1}{t^2} \begin{bmatrix} 1 & 2t \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2t & -t^2 \\ -1 & t \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{bmatrix}$$

p. 409 / Sect. 7.6 / #5 Find the general solution of

$$\underline{x}' = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix} \underline{x}$$

$A = \begin{pmatrix} 1 & -1 \\ 5 & -3 \end{pmatrix}$; $\quad p_A(s) = s^2 + 2s + 2 = (s+1)^2 + 1$

Hence $\quad s_{1,2} = -1 \pm i$

$s_1 = -1 + i$: $\quad \left[A + (1-i)I\right] \cdot \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \iff \begin{bmatrix} 2-i & -1 \\ 5 & -2-i \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$

$(2-i)a - b = 0 \Rightarrow \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ (2-i)a \end{bmatrix} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix} a$

$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 2-i \end{bmatrix}$: $\quad e^{(-1+i)t} \begin{bmatrix} 1 \\ 2-i \end{bmatrix} = e^{-t} \cdot (\cos(t) + i \sin(t)) \begin{bmatrix} 1 \\ 2-i \end{bmatrix} =$

$$= e^{-t} \begin{bmatrix} \cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} + i\, e^{-t} \begin{bmatrix} \sin(t) \\ 2\sin(t) - \cos(t) \end{bmatrix}$$

$$\Rightarrow \underline{x}(t) = c_1 e^{-t} \begin{bmatrix} \cos(t) \\ 2\cos(t) + \sin(t) \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} \sin(t) \\ 2\sin(t) - \cos(t) \end{bmatrix}$$

$\boxed{\text{p.516 / Sect. 9.3 / \#6}}$

$$\begin{cases} \dfrac{dx}{dt} = x - x^2 - xy \\[2mm] \dfrac{dy}{dt} = 3y - xy - 2y^2 \end{cases}$$

(a) Critical Points:

$$\begin{cases} x - x^2 - xy = 0 \\ 3y - xy - 2y^2 = 0 \end{cases} \iff \begin{cases} x(1-x-y) = 0 \\ y(3-x-2y) = 0 \end{cases} \iff \begin{cases} (x=0) \ \underline{\text{or}} \ (1-x-y=0) \\ (y=0) \ \underline{\text{or}} \ (3-x-2y=0) \end{cases}$$

$$\iff \begin{cases} x=0 \\ y=0 \end{cases} \ \underline{\text{or}} \ \begin{cases} x=0 \\ 3-x-2y=0 \end{cases} \ \underline{\text{or}} \ \begin{cases} 1-x-y=0 \\ y=0 \end{cases} \ \underline{\text{or}} \ \begin{cases} 1-x-y=0 \\ 3-x-2y=0 \end{cases}$$

$$\iff P_1(0,0), \quad P_2\left(0, \tfrac{3}{2}\right), \quad P_3(1,0), \quad P_4(-1,2).$$

(b&c) Linearization at the 4 critical points:

$$J(x,y) = \begin{bmatrix} 1-2x-y & -x \\ -y & 3-x-4y \end{bmatrix}$$

The Local Linear System: $\begin{pmatrix} u \\ v \end{pmatrix}' = J(P) \cdot \begin{pmatrix} u \\ v \end{pmatrix}$

$J_1 = J(P_1) = J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \Rightarrow$ unstable node, $\lambda_1 = 1$, $\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\lambda_2 = 3$, $\underline{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

LLS: $\begin{cases} u' = u \\ v' = 3v \end{cases}$

$J_2 = J(P_2) = J(0, \tfrac{3}{2}) = \begin{bmatrix} -1/2 & 0 \\ -3/2 & -3 \end{bmatrix}$: asymptotically stable node, $\lambda_1 = -\frac{1}{2}$, $\underline{v}^{(1)} = \begin{bmatrix} 1 \\ -\frac{3}{5} \end{bmatrix}$; $\lambda_2 = -3$, $\underline{v}^{(2)} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

LLS: $\begin{cases} u' = -\frac{1}{2}u \\ v' = -\frac{3}{2}u - 3v \end{cases}$

$J_3 = J(P_3) = J(1,0) = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$: unstable saddle, $\lambda_1 = -1$, $\underline{v}^{(1)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\lambda_2 = 2$; $\underline{v}^{(2)} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

LLS: $\begin{cases} u' = -u - v \\ v' = 2v \end{cases}$

$J_4 = J(P_4) = J(-1,2) = \begin{bmatrix} 1 & 1 \\ -2 & -4 \end{bmatrix}$: $p_4(s) = s^2 + 3s - 2 = 0$; $\lambda_{1,2} = \dfrac{-3 \pm \sqrt{17}}{2}$ $\begin{cases} \to + \ \lambda_1 \\ \to - \ \lambda_2 \end{cases}$: unstable saddle.

LLS: $\begin{cases} u' = u + v \\ v' = -2u - 4v \end{cases}$

$\underline{v}^{(1)} = \begin{bmatrix} 1 \\ -\frac{5}{2} + \frac{\sqrt{17}}{2} \end{bmatrix}$, $\underline{v}^{(2)} = \begin{bmatrix} 1 \\ -\frac{5}{2} - \frac{\sqrt{17}}{2} \end{bmatrix}$

(d)