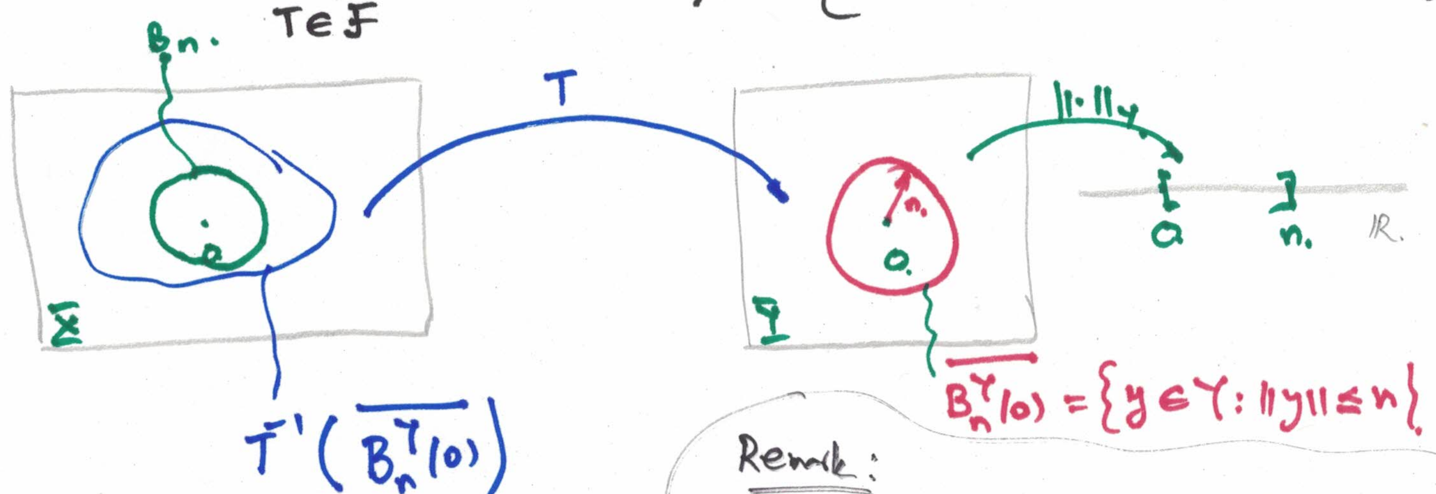
Continue with the Uniform Boundedness Principle:

Recap:

Theorem: Let $(X, \|\cdot\|)$ be a Banach space. Let $\mathcal{F} \subset B(X,Y)$ where $(Y, \|\cdot\|_Y)$ is a normed linear space. Suppose for each $x \in X$, $\{\|Tx\|_Y, T \in \mathcal{F}\} \subset \mathbb{R}$ is bounded. Then $\{\|T\|, T \in \mathcal{F}\}$ is bounded

Proof.

Idea: let $B_n = \bigcap_{T \in \mathcal{F}} T^{-1}\left(\overline{B_n^Y(0)}\right) = \left\{x \in X : \|Tx\|_Y \le n, \forall T \in \mathcal{F}\right\}$



Remark:
closure$(\{y \in Y : \|y\| < r\}) = \{y \in Y : \|y\| \le r\}$

① $B_n = \bigcap_{T \in \mathcal{F}} T^{-1}\left(\overline{B_n^Y(0)}\right)$

1. Each T continuous $\rightarrow$ $T^{-1}\left(\overline{B_n^Y(0)}\right)$ is closed.
2. $B_n = \bigcap_{T \in \mathcal{F}}$ closed sets is closed.
3. $0 \in B_n$ $\rightarrow$ [Each B_n is a non-empty closed set.

② Claim: $X = \bigcup_{n \ge 1} B_n$.

Why the claim: $\bigcup_n B_n = \overline{X}$: (2)

Take $x \in \overline{X}$ $\dashrightarrow$ $\{\|Tx\|, T \in \mathcal{F}\} \subset [0,N]$, for some $N \in \mathbb{N}$

$\dashrightarrow$ $x \in B_N$.

(3) $\overline{X}$ complete.

$\left.\begin{array}{l} \overline{X} \text{ complete.} \\ \overline{X} = \bigcup_{n \geq 1} B_n \\ B_n \text{ closed.} \end{array}\right\} \xRightarrow[\text{Theorem.}]{\text{By Baire category}} \exists N$ s.t. B_N has nonempty interior.

$$\Rightarrow \exists \underset{r_0>0}{x_0 \in \overline{X}} : \quad B_{r_0}(x_0) \subset B_N = \bigcap_{T \in \mathcal{F}} T^{-1}\left(\overline{B^Y_N(0)}\right)$$

$$\Rightarrow B_{r_0}(x_0) \subset T^{-1}\left(\overline{B^Y_N(0)}\right)$$

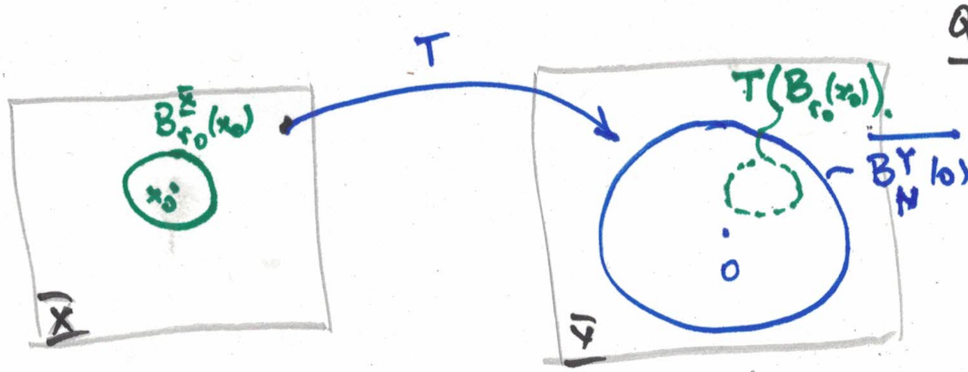
Scale: $\longrightarrow$

$$B_{\frac{r_0}{N}}\left(\frac{x_0}{N}\right) \subset T^{-1}\left(\overline{B^Y_1(0)}\right).$$

By previous proposition: $\|T\|_{B(\overline{X},Y)} \leq \dfrac{1 + \|T\frac{x_0}{N}\|_Y}{\frac{r_0}{N}} = \dfrac{N + \|Tx_0\|_Y}{r_0}$

$$\sup_{T \in \mathcal{F}} \|T\|_{B(\overline{X},Y)} \leq \frac{N + \sup_{T \in \mathcal{F}} \|Tx_0\|_Y}{r_0} \leq \frac{2N}{r_0}$$

QED.



<u>Corollary</u> [(RS) III.5] Let $\underline{X}$ and $\underline{Y}$ be Banach spaces and let $B(\cdot,\cdot)$ be a separately continuous bilinear mapping from $\underline{X}\times\underline{Y}$ to $\mathbb{C}$, that is, for each fixed x, $B(x,\cdot): Y \to \mathbb{C}$ is a bounded linear transformation, and for each fixed y, $B(\cdot,y): \underline{X} \to \mathbb{C}$ is also a bounded linear transformation. Then $B(\cdot,\cdot)$ is jointly continuous, that is, if $x_n \to 0$ and $y_n \to 0$ then $B(x_n,y_n) \to 0$. Even stronger, there is $M>0$ s.t. $|B(x,y)| \le M\cdot\|x\|\cdot\|y\|$. ⌋.

<u>Proof</u>:

Fix $(x_n)_n, (y_n)_n$: $x_n \to 0$, $y_n \to 0$.

Let $T_n(y) = B(x_n, y)$. $T_n : Y \to \mathbb{C}$.

Each T_n is bounded: ~~$\{\|T_n(y)\|\}$, is bdd~~

Fix y $\to$ $\lim_{n\to\infty} T_n(y) = \lim_{n\to\infty} B(x_n,y) = 0$.

$x \mapsto B(x,y)$ is bounded: $|B(x,y)| \le C_y \cdot \|x\|$

by continuity with y fixed.

$\Rightarrow \{\underline{\|T_n(y)\|,\ n\ge 1}\}$ is bounded, $\forall \underline{\underline{y}}$.

By uniform boundedness principle. $\longrightarrow$ $\sup_n \|T_n\| < \infty$: $\longrightarrow$ $|T_n(y)| \le C\cdot\|y\|$, $\forall n, \forall y$

$|T_n(y)| \le \|T_n\|\cdot\|y\|$.

$|B(x_n,y)| \le C.\|y\|$, $\forall n, \forall y$.

Then $|B(x_n,y_n)| \le C\cdot\|y_n\| \Rightarrow \lim_{n\to\infty} |B(x_n,y_n)| = 0$. ⌋.

<u>Remark</u> If B is not linear, then conclusion is false:

$F : \mathbb{R}\times\mathbb{R}$, $\quad F(x,y) = \begin{cases} \dfrac{x\cdot y}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0). \end{cases}$

Fix $x\neq 0$: $\quad y \longmapsto \dfrac{xy}{x^2+y^2} = \dfrac{\frac{y}{x}}{1+\left(\frac{y}{x}\right)^2}$: continuous.

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For $x=0$: $\quad y \longmapsto 0$ continuous.

$y=0$: $\quad x \longmapsto 0$ continuous.

Jointly: $\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta). \end{aligned}$: $\quad F(r\cos(\theta), r\sin(\theta)) = \dfrac{r^2\sin(\theta)\cos(\theta)}{r^2} =$

$$= \sin(\theta)\cos(\theta) = \frac{1}{2}\sin(2\theta).$$

$$\lim_{r\to 0} F(r\cos(\theta), r\sin(\theta)) = \frac{1}{2}\sin(2\theta) \neq 0.$$

Theorem [Open Mapping Theorem] let $T: X \to Y$ be a bounded linear transformation from $(\underline{X}, \|\cdot\|_{\underline{X}})$ a Banach space, to $(\underline{Y}, \|\cdot\|_{Y})$ also a Banach space. Assume T is onto (surjective), i.e. $T(\underline{X}) = \underline{Y}$. Then T is an open map, i.e. for any $E \subset \underline{X}$ an open set, $T(E)$ is open in $\underline{Y}$.

Proof

Assume that we showed that $\overline{T(B_r^{\underline{X}}(0))}$ has a non empty interior for some r.

Claim: If this assumption is true, then for any open set $E \subset \underline{X}$, $T(E)$ is open in Y.

Proof of this claim:

$$\text{let } \underbrace{y \in T(E)} \longrightarrow \exists x \in E: \ T(x) = y \in T(E).$$

$$E \text{ open} \dashrightarrow \exists r_1 : B_{r_1}(x) \subset E. \longrightarrow T(B_{r_1}(x)) \subset T(E).$$

$$T(E) \supset T(B_{r_1}(x)) = T(x) + T(B_{r_1}(0)) = T(x) + \frac{r_1}{r} \cdot \underbrace{T(B_r(0))}_{\text{non empty interior.}}.$$

$\Rightarrow \exists A \subset Y$ open $: 0 \in A \subset T(B_r(0))$

$A \neq \phi$

$0 \in A$

$$\Rightarrow \underbrace{T(x) + \frac{r_1}{r} \cdot A}_{\text{open. and. includes } y.} \subset T(E). \longrightarrow \boxed{T(E) \text{ open}}$$

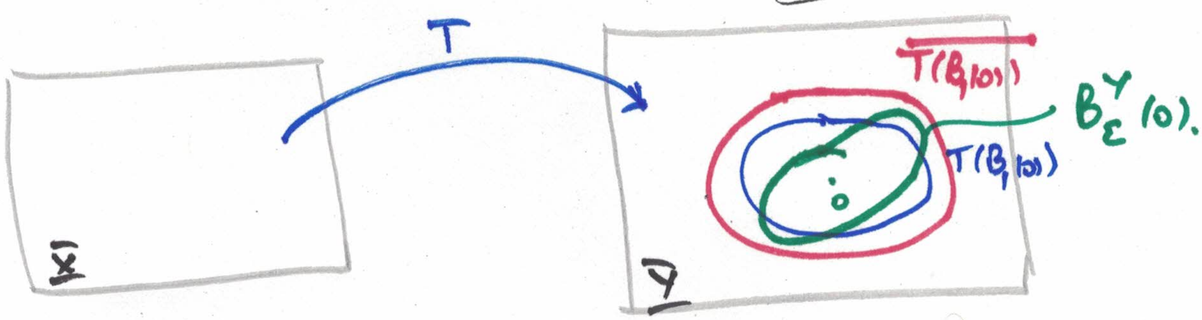
Remains to prove: $\exists\, r>0$ s.t. $T(B_r(0))$ has non empty interior.

T surjective $\Rightarrow$ $$\bigcup_{n\geq 1} T(B_n(0)) = Y.$$

By Baire category theorem: $\exists\, n$: $\overline{T(B_n(0))}$ has a nonempty interior.

$\xrightarrow[\text{divide by } n]{\text{by linearity}}$ $\overline{T(B_1(0))}$ has a non empty interior.

: Find $\varepsilon > 0$ s.t. $\boxed{B^Y_\varepsilon(0) \subset \overline{T(B_1(0))}}$



Claim 2: $\overline{T(B_1(0))} \subset T(B_2(0))$.

Assume Claim 2 is proved: $B^Y_\varepsilon(0) \subset \overline{T(B_1(0))} \subset T(B_2(0))$.

$\Rightarrow B^Y_{\frac{\varepsilon}{2}}(0) \subset T(B_1(0)) \Rightarrow T(B_1(0))$ has non-empty interior.

Let $y \in \overline{T(B_1(0))}$. $\exists\, x_1 \in B_1(0)$: $\| T(x_1) - y \| < \frac{\varepsilon}{2}$.

$$\Rightarrow y - T(x_1) \in B^Y_{\varepsilon/2}(0) \underset{\substack{\uparrow \\ \text{by scaling } B^Y_\varepsilon(0) \subset \overline{T(B_1(0))}}}{\subset} \overline{T(B_{1/2}(0))}$$

There exists $x_2 \in B^{X}_{1/2}(0)$:

s.t. $\| y - T(x_1) - T(x_2) \| < \frac{\varepsilon}{4}$.

$$y - T(x_1 + x_2) \in B^{Y}_{\varepsilon/4}(0) \subset \overline{T(B_{1/4}(0))}$$

Construct $x_3, x_4, \ldots$:

$$\| y - T(\underbrace{x_1 + x_2 + \cdots + x_n}) \| < \frac{\varepsilon}{2^n}.$$

each. $\|x_k\| \leq \frac{1}{2^{k-1}}$

because X is complete.

$$\Rightarrow x = \sum_{k=1}^{\infty} x_k \text{ is convergent in } X.$$

$$\| y - T(x) \| = \lim_{n \to \infty} \| y - T(x_1 + \cdots + x_n) \| = 0.$$

$$\Rightarrow y = T(x). \qquad : \|x\| \leq \sum_{k=1}^{\infty} \|x_k\| \leq \sum_{k=1}^{\infty} \frac{1}{2^{k-1}} = 2.$$

$$\Rightarrow x \in \overline{B_2(0)}.$$

$$\Rightarrow y \in T(\overline{B_2(0)}) \subset T(B_{2.1}(0)).$$

Therefore: $\overline{T(B_1(0))} \subseteq T(B_{2.1}(0))$.

$\blacksquare$