Compact Operators

__Theorem.__ Let $\underline{X}$ and $\underline{Y}$ be two Banach spaces.

(a) If $T \in B(\underline{X},\underline{Y})$ and $\dim \operatorname{Ran}(T) < \infty$ then T is compact.

(b) If $T \in B(\underline{X},\underline{Y})$ is compact and $\operatorname{Ran}(T)$ is closed in $\underline{Y}$, then $\dim \operatorname{Ran}(T) < \infty$.

(c) If $T \in B(\underline{X})$ is compact and $\lambda \in \mathbb{C}$, $\lambda \neq 0$. then

$$\dim \ker(\lambda I - T) < \infty.$$

(d) If $T \in B(\underline{X})$ is compact and $\dim(\underline{X}) = \infty$ then $0 \in \sigma(T)$.

(e) If $\underbrace{S \in B(\underline{X})}_{\substack{\text{bounded} \\ \text{not necessarily compact}}}$, and $\underbrace{T \in B(\underline{X}) \text{ is compact}}_{\substack{T \text{ bounded and} \\ \text{Compact}}}$ then ST and TS are compact.

(f) The set of compact operators of $B(\underline{X},\underline{Y})$ forms a closed subspace of $B(\underline{X},Y)$ in its norm topology. In other words, if $K(\underline{X},Y)$ denotes the set of compact operators between $\underline{X}$ and $\underline{Y}$ then the closure of $K(\underline{X},\underline{Y})$ w.r.t. operator norm is $K(X,Y)$ itself.

__Remark:__ (e) & (f) $\Rightarrow$ $K(\underline{X}) \subset B(\underline{X})$ is a two-sided closed ideal in $B(\underline{X})$.

Whenever $I \subset A$ is a closed two-sided ideal in ~~a~~ Banach algebra A.

One can construct A/I also a Banach algebra.

Here: $B(X)/K(X)$ is called the Calkin algebra.

Proof. Let $U = \overline{B_1(0)} = \{x \in X : \|x\| \leq 1\}$. be the closed unit ball.

(a) $\longrightarrow$ $\overline{T(U)}$: closed & bounded set in a finite dimensional normed vector space.

$\longrightarrow$ $\overline{T(U)}$ is compact $\Rightarrow$ T is compact.

(b)

$$T : X \longrightarrow Z, \quad Z = \mathrm{Ran}(T),$$

$(Z, \|\cdot\|_Y)$ is a Banach space.

T is onto (surjective) $\xRightarrow[\text{mapping theorem.}]{\text{by the open}}$ $T(B_1(0))$ is open.

$$0 \in \underbrace{T(B_1(0))}_{\text{open}} \subset T(\overline{B_1(0)}) \subset \underbrace{\overline{T(\overline{B_1(0)})}}_{\text{compact.}} = K$$

$\Rightarrow$ 0 has a compact neighborhood. $\dashrightarrow$ $\exists$ $B_r(0) \subset K$ compact

$\Rightarrow$ $\overline{B_r(0)}$ compact.

$\longrightarrow$ $\dim Z < \infty$. : $\dim \mathrm{Ran}(T) < \infty$.

(c).

Assume $T : \underline{X} \to \underline{X}$ is compact and $\lambda \in \mathbb{C}$; $\lambda \neq 0$.

Let $V = \ker(\lambda I - T) \subset \underline{X}$.

T bounded $\dashrightarrow$ $\lambda I - T$ is bounded $\dashrightarrow$ V is closed space.

$\| Tx \| \leq \|T\| \cdot \|x\|$. $\longrightarrow$ $\cancel{\|x_n - x_m\|}$

$(x_n)_n$ Cauchy, $x_n \in V$

$x_n \to x$, in $\underline{X}$ $\xrightarrow{\lambda I - T \text{ bounded.}}$ $(\lambda I - T)(x) = \lim\limits_{n\to\infty} (\lambda I - T)(x_n) = 0$.

$T|_V : V \to \underline{X}$, $\longrightarrow$ $T|_V(x) = \lambda \cdot x$.

$\| T|_V(x) \| \geq |\lambda| \cdot \|x\|$.

$\longrightarrow$ $\mathrm{Ran}(T|_V)$ is closed.

$(x_n)_n \in V$: $(T_V(x_n))_n$ Cauchy. $\longrightarrow$ $(x_n)_n$ is Cauchy.

$\downarrow$

Convergent in V too

$\downarrow$

$T|_V(x_n) \to T|_V(x) \in \mathrm{Ran}(T|_V)$.

$T|_V : V \to \underline{X}$ is still compact.

$\downarrow$

$T|_V : V \to \mathrm{Ran}(T|_V)$ compact.

by (b) $\to \dim \underbrace{\mathrm{Ran}(T|_V)}_{V} < \infty$. $\Rightarrow \dim V < \infty$.

(d) Assume $T: X \to X$ is compact

and $\dim X = \infty$.

Assume $0 \notin \sigma(T) \dashrightarrow 0 \in \rho(T)$. (resolvent set). $\dashrightarrow$

$\to$ T is invertible $\dashrightarrow$. $\exists\, S \in B(X)$ s.t. $S \cdot T = 1_X$

(identity).

closed set.

$\downarrow$

$$U \xrightarrow{T} T(U) \subset \underbrace{\overline{T(U)}}_{\text{compact.}} \xrightarrow{S} \underline{S(\overline{T(U)})}_{\text{compact.}}$$

$$U = \overline{B_1(0)}$$

$$T(U) \xrightarrow{S} \overline{S(T(U))} \text{ compact set.}$$

$$U \xrightarrow{S \cdot T} ST(U) = 1_X(U) = U \text{ closed.}$$

$\Rightarrow$ U is compact.

$\dim X = \infty$. $\not\to$ contradiction.

(e) Assume. $S \in B(X)$, $T \in B(X)$, $\underline{T \text{ is compact}}$. | Want: $S.T$ compact

Proceed as above:

$$U = \overline{B_1(0)} \xrightarrow{T} T(U) \subset \underbrace{\overline{T(U)}}_{\text{compact}} \xrightarrow{S} \underbrace{S(\overline{T(U)})}_{\text{compact.}}.$$

$$U \xrightarrow{ST} ST(U) \subset S(\overline{T(U)})$$

$$\Rightarrow \overline{ST(U)} \subset S(\overline{T(U)}) \text{ compact}$$

$\Rightarrow \overline{ST(U)}$ is compact $\Rightarrow ST$ is compact.

Similarly $T \cdot S$.

(f).

11. Let $K(X,Y)$ denote the set of compact operators from X to Y.

Claim: $K(X,Y)$ is a linear subspace of $B(X,Y)$.

Pf. $T_1, T_2 \in K(X,Y)$.

$\overline{(T_1+T_2)(U)}$ is compact in Y.

when we know $\underbrace{\overline{T_1(V)}}_{Z_1}$, $\underbrace{\overline{T_2(V)}}_{Z_2}$ are compact.

$\overline{(T_1+T_2)(U)} \subset \underbrace{Z_1 + Z_2}$.

Subclaim: $Z_1 + Z_2$ is compact.

Why: Take. $(x_n+y_n)_n$; $(x_n)_n \subset Z_1$, $(y_n)_n \subset Z_2$.

Z_1, Z_2 are sequentially compact.

$\rightarrow \exists (x_{n_k})_k, (y_{n_k})_k$ convergent subsequences.

$\rightarrow (x_{n_k}+y_{n_k})_k$ convergent in $Z_1 + Z_2$.

$\rightarrow Z_1 + Z_2$ is sequentially compact $\rightarrow Z_1 + Z_2$ compact.

$T_1 + T_2$ compact

(2) claim: $K(\underline{X},Y)$ is closed in $B(\underline{X},Y)$.

Let. $T \in$ closure $K(\underline{X},Y)$ in $B(\underline{X},Y)$.

Want: $T(U)$, is totally bounded: $\forall \varepsilon > 0 \ \exists N_\varepsilon, x_1 \dots x_{N_\varepsilon} \in U$

s.t. $T(U) \subset B_\varepsilon(Tx_1) \cup \dots \cup B_\varepsilon(Tx_{N_\varepsilon})$

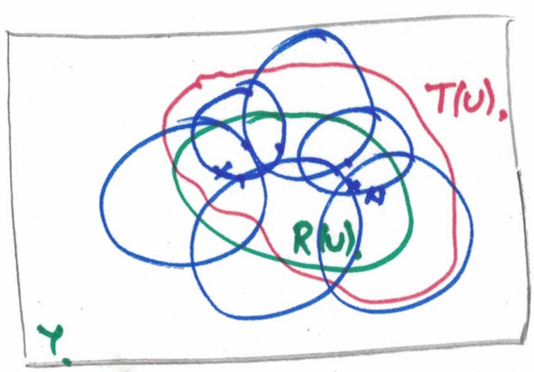
Recall: If. $A \subset (\underline{X}, d)$ ← metric space is: (i) totally bounded. (ii) A complete. $\Longrightarrow$ A is compact.

$\boxed{\text{Fix } \varepsilon > 0}$

Let $R \in K(\underline{X},Y)$ be a compact operator s.t. $\|T - R\| < \frac{\varepsilon}{3}$

Since. $R(U)$ is totally bounded. $\longrightarrow \exists x_1 \dots x_N \in U$

s.t. $R(U) \subset \bigcup_{k=1}^{N} B_{\varepsilon/3}(Rx_k)$



claim: $T(U) \subset \bigcup_{k=1}^{N} B_\varepsilon(Tx_k)$

Take $x \in U \dashrightarrow \|Tx - Rx\| < \frac{\varepsilon}{3}$

$\dashrightarrow$ Find. x_{k_0} s.t. $\|Rx - Rx_{k_0}\| < \frac{\varepsilon}{3}$. $\Longrightarrow$

$\dashrightarrow \|Tx_{k_0} - Rx_{k_0}\| < \frac{\varepsilon}{3}$.

$$\| Tx - Tx_{k_0} \| \le \| Tx - Rx \| + \| Rx - Rx_{k_0} \| + \| Rx_{k_0} - Tx_{k_0} \| < \varepsilon$$

$$\Rightarrow \quad Tx \in B_{\varepsilon}(Tx_{k_0}).$$

$$\Rightarrow \quad \underbrace{T(U) \subset \bigcup_{k=1}^{N} B_{\varepsilon}(Tx_k)}.$$

Finite cover.

$\longrightarrow$ $T(U)$ is totally bounded. $\longrightarrow$ $\overline{T(U)}$ compact.

$\longrightarrow T \in K(X, Y)$.

Recall the following Theorem:

<u>Theorem</u> [Ascoli's Theorem] Suppose (X, d) is a compact metric space, let $C(X)$ denote the sup-norm Banach space of complex-valued continuous functions over X. Assume $\Phi \subset C(X)$ satisfies:

(1) (pointwise boundedness) $\forall x \in X$, $\{ |f(x)|, f \in \Phi \}$ is bounded in $\mathbb{R}$

(2) (equicontinuity)

$\forall \varepsilon > 0 \ \exists \delta > 0$, $\forall x, y \in X$, $\forall f \in \Phi$, $d(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$.

Then Φ is totally bounded in $C(X)$, and therefore $\overline{\Phi}$ (the closure) is compact in $C(X)$

Want:

Theorem. Assume. H is a Hilbert space, and $T \in B(H)$ is a compact operator. Then T^* is also compact.

Remark: If X, Y are Banach spaces and $T \in B(X,Y)$ is compact Then. T^t (the transpose, or dual operator), $T^t \in B(Y^*, X^*)$ is also compact, $\left[\; T^t(\ell)(x) = \ell(Tx). \; \right]$.
$\ell \in Y^*, x \in X$

Proof.

Assume T is compact. Let $(y_n)_n \in H \sim H^*$ (dual. by Riesz rep. th.)

$\|y_n\| = 1.$

Want: extract from $(T^* y_n)_n$ a convergent subsequence.

Let $\ell_n = T^* y_n$, $\ell_n(x) = \langle x, T^* y_n \rangle = \langle Tx, y_n \rangle$

$\ell_n \in H^* \sim H.$

Claim: $\{\ell_n, n \geq 1\} \subset C(\underbrace{\overline{T(U)}}_{X}^{\text{compact set}})$. is equicontinuous. & uniformly bounded.

$\xrightarrow[\text{Theorem}]{\text{Ascoli's}}$ closure is compact $\dashrightarrow \exists (n_k)_k : (\ell_{n_k})_k$ convergent in H-norm.

$$\underline{X} = \overline{T(U)} \longrightarrow \text{compact set in } H.$$

↳ closed unit ball in H.

Why the claim:

(1) Boundedness:

$$\forall x \in \underline{X} : \quad |\ell_n(x)| = |\langle Tx, y_n\rangle| \leq \|Tx\| \cdot \|y_n\| = $$
$$= \|Tx\| < \infty.$$
$$\longrightarrow \{|\ell_n(x)|\}_{n\geq 1} \text{ bounded.}$$

(2) Equicontinuous:

$$|\ell_n(x) - \ell_n(y)| = |\langle Tx, y_n\rangle - \langle Ty, y_n\rangle| =$$
$$= |\langle T(x-y), y_n\rangle| \leq \|T\| \cdot \|x-y\| \cdot \underbrace{\|y_n\|}_{1}$$

$$\dashrightarrow \forall \varepsilon > 0, \text{ choose } \delta = \frac{\varepsilon}{2\|T\|} \longrightarrow |\ell_n(x) - \ell_n(y)| < \varepsilon.$$
$$\forall x, y \in \underline{X}, \ \|x-y\| < \delta, \ \forall n \geq 1.$$

By Ascoli's theorem: $\overline{\{\ell_n\}}$ is compact in H $\longrightarrow$ extract convergent subsequence.

$$(T^* y_{n_k})_{n_k} \text{ is convergent} \Longrightarrow T^* \text{ is compact.}$$

$$\Rightarrow \overline{T^*(U)} \text{ compact} \nearrow$$
