L14

Compact Operators (2)

From now on consider the setup of Hilbert spaces, H.

Additional assumption: H is separable. $\longleftrightarrow$ H admits a countable ORTHONORMAL BASIS.

Theorem A compact operator maps weakly convergent sequences into norm convergent sequences.

Proof.

Assume $T \in K(H)$, i.e. $T : H \to H$ compact.

Assume $x_n \xrightarrow{w} x$, in H : $\forall \ell \in H^* \simeq H$: $\lim_{n\to\infty} \langle \ell, x_n - x \rangle = 0$

Claim (what we need to show):

$$Tx_n \longrightarrow Tx \text{, in } H: \quad \lim_{n\to\infty} \| Tx_n - Tx \| = 0.$$

Proof by contradiction:

Assume. $Tx_n \not\to Tx$ ⋯→ There exists $\varepsilon_0 > 0$ and

a subsequence. $(x_{n_k})_{k\geq 1}$ s.t. $\| Tx_{n_k} - Tx \| \geq \varepsilon_0$.

T being compact: $\overline{T(U)}$ is compact.

Sequence $\{ \|x_n\| \}_{n\geq 1}$ is bounded: by uniform boundedness principle.

let $R > 0$, $\|x_n\| \leq R$, $\forall n$. ⋯→ $\overline{T(U_R)}$ is compact in H.

But $\left(Tx_{n_k}\right)_k \subset \overline{T(U_R)}$

compact in H.

$\longrightarrow \exists$ subsubsequence: $\left(Tx_{n_{k_j}}\right)_j$ convergent in H.

let $z = \lim_{j\to\infty} Tx_{n_{k_j}}$ $\dashrightarrow$ $Tx_{n_{k_j}} \xrightarrow{w} z$

$z = Tx$

~~we~~ Supposedly: $Tx_{n_{k_j}} \xrightarrow{w} Tx$. $\longrightarrow$ Contradiction

and $\|z - Tx\| \geq \varepsilon_0$.

$\boxtimes$

Uniqueness of weak limit:

$\forall \ell$: ~~lim ⟨ℓ, x_n, z⟩~~ $\lim_{n\to\infty} \langle \ell, x_n - z_1\rangle = 0$

$\lim_{n\to\infty} \langle \ell, x_n - z_2\rangle = 0$

$\lim_{n\to\infty} \langle \ell, z_1 - z_2\rangle = 0.$

$\langle \ell, z_1 - z_2\rangle = 0$: $\ell = z_1 - z_2$ $\dashrightarrow$ $\|z_1 - z_2\| = 0.$

Theorem. [Finite Approximation Property]. Assume H is a separable Hilbert space and $T \in K(H)$ is a compact operator. Then for any $\varepsilon > 0$ there exists T_ε a finite rank operator s.t.

$$\| T - T_\varepsilon \| < \varepsilon.$$

Proof.

Let $\{e_n\}_{n \geq 1}$ be an ONB.

Construct inductively: $\lambda_0 = \sup_{\|x\|=1} \|Tx\|$,

$$\lambda_n = \sup_{\substack{\|x\|=1 \\ x \perp \{e_1, e_2, \dots, e_n\}}} \|Tx\|$$

$\to$ $\lambda_1 \geq \lambda_2 \geq \dots$ monotone decreasing sequence.

Take. $\psi_n : \|\psi_n\| = 1$, $\psi_n \perp \{e_1, e_2, \dots, e_n\}$:

$$\|T\psi_n\| \geq \frac{\lambda_n}{2}$$

Claim: $(\psi_n)_{n \geq 1}$ converges weakly to 0:

Fix $y \in H$. Want: $\lim_{n\to\infty} \langle y, \psi_n \rangle = 0$.

$\downarrow$ Fix $\varepsilon > 0$ $\exists N$: $|\langle y, \psi_n \rangle| < \varepsilon$, $\forall n \geq N$.

$y = \sum^{\infty} y_i e_i$, $y_i = \langle y, e_i \rangle$. : $\langle y, \psi_n \rangle = \sum^{\infty} y_i \langle e_i, \psi_n \rangle$

$$|\langle y, \psi_n\rangle| = \left|\sum_{j=n+1}^{\infty} y_j \langle e_j, \psi_n\rangle\right| \le \left(\sum_{j=n+1}^{\infty} |y_j|^2\right)^{1/2} \underbrace{\left(\sum_{j=n+1}^{\infty} |\langle e_j, \psi_n\rangle|^2\right.}_{\le \sum_{j=1}^{\infty} |\langle e_j, \psi_n\rangle|^2 =}$$

$$\le \left(\sum_{j=n+1}^{\infty} |y_j|^2\right)^{1/2}.$$

Find N: $\left(\sum_{j=N}^{\infty} |y_j|^2\right)^{1/2} < \varepsilon \longrightarrow \forall n \ge N : |\langle y, \psi_n\rangle| < \varepsilon.$

Consequence: $\psi_n \xrightarrow{w} 0$: ① $T\psi_n \longrightarrow 0$ in H norm.

$$0 < \frac{\lambda_n}{2} \le \|T\psi_n\|$$

$$\Rightarrow \lim_{n\to\infty} \lambda_n = 0.$$

②. let

$$T_n(x) = \sum_{k=1}^{n} \langle x, e_k\rangle T e_k. \dashrightarrow \text{finite rank.}$$

$\dim \operatorname{Ran} T_n \le \dim \{Te_1, \ldots$

$\le n.$

$$(T - T_n)(x) = \sum_{k=1}^{\infty} \langle x, e_k\rangle T e_k - \sum_{k=1}^{n} \langle x, e_k\rangle T e_k$$

$$x = \sum_{k=1}^{\infty} \langle x, e_k\rangle e_k \qquad = \sum_{k=n+1}^{\infty} \langle x, e_k\rangle T e_k.$$

$$\|T - T_n\| = \sup_{\|x\|=1} \left\| \sum_{k=n+1}^{\infty} \langle x, e_k \rangle T e_k \right\| =$$

$$x = x^{\parallel} + x^{\perp}, \qquad x^{\parallel} \in \operatorname{span}\{e_1 \dots e_n\}.$$

$$\|x\|^2 = \|x^{\parallel}\|^2 + \|x^{\perp}\|^2 \qquad x^{\perp} \in \overline{\operatorname{span}}\{e_{n+1}, \dots \}.$$

$$\sum_{k=n+1}^{\infty} \langle x, e_k \rangle T e_k = \sum_{k=n+1}^{\infty} \langle x^{\perp}, e_k \rangle T e_k$$

$$= \sup_{\substack{\|x\|=1 \\ x \perp \{e_1 \dots e_n\}}} \Big\| \underbrace{\sum_{k=n+1}^{\infty} \langle x, e_k \rangle T e_k}_{T\left(\underbrace{\sum_{k=n+1}^{\infty} \langle x, e_k \rangle e_k}_{x}\right)} \Big\| = \sup_{\substack{\|x\|=1 \\ x \perp \{e_1 \dots e_n\}}} \|Tx\| = \lambda_n$$

Thus: $\|T - T_n\| = \lambda_n$.

and we showed earlier: $\lambda_n \downarrow 0$

$$\Rightarrow \lim_{n \to \infty} T_n = T, \text{ in uniform topology}$$

(operator norm topology).

$\square$

Matrix Representation :

$\{e_1, e_2, \dots\}$ ONB in H. , $T : H \to H$.

$$T: \quad \begin{array}{c|cccc} & e_1 & e_2 & e_3 & \dots e_n. \\ \hline e_1 & \langle Te_1, e_1\rangle & & & \\ e_2 & \langle Te_1, e_2\rangle & & & \\ e_3 & \vdots & & & \\ \vdots & & & & \\ & Te_1. & Te_2 & \dots & \end{array}$$

$$T_n : \quad \begin{bmatrix} T_n. & 0 \end{bmatrix}$$

$$\lim_{n\to\infty} \left\| \left[\, 0 \; \dots \; 0 \;\middle|\; Te_{n+1} \;\middle|\; Te_{n+2} \;\middle|\; \dots \,\right] \right\|_{2\to 2} = 0.$$

Theorem [Special Case of Banach - Alaoglu Theorem].

Assume H is a separable Hilbert space. Then $U = \{x \in H : \|x\| \le 1\}$ is weakly compact.

Remark: Banach - Alaoglu Theorem: If $(X, \|\cdot\|)$ is a Banach space then $U = \{ \cancel{x \in} X : \|x\| \le 1\}$ is weak* compact.

Proof.

Need to show: If. $(x_n)_{n \ge 1}$ is a sequence in H, $\|x_n\| \le 1$.

then there exists a subsequence $(x_{n_k})_{k \ge 1}$ ~~and~~. $z \in H$

such that for every $l \in H$, $\lim_{k \to \infty} \langle l, x_{n_k} - z \rangle = 0$.

Why sequential compactness: Use Cantor diagonal argument (trick):

$$x_n = \sum_{j=1}^{\infty} x_{n,j}\, e_j \quad , \qquad x_{n,j} = \langle x_n, e_j \rangle \in \mathbb{C}.$$

$$
\begin{array}{ccccc}
 & e_1 & e_2 & \cdots & \\
x_1 \to & x_{1,1} & x_{1,2} & x_{1,3} \cdots & \text{coordinates of vector } x_1 \\
x_2 \to & x_{2,1} & x_{2,2} & x_{2,3} \cdots & \\
 & x_{3,1} & \vdots & & \\
 & \vdots & \vdots & & \\
x_n \to & x_{n,1} & & & \\
 & \vdots & & & \\
 & \underbrace{}_{1^{st} \text{ coordinate of } \{x_1, x_2 \ldots\}.} & & &
\end{array}
$$

All $|x_{n,j}| \leq 1$. $\longrightarrow$ Extract a subsequence $(n_k^1)_{k \geq 1}$

s.t. $\left(x_{n_k^1, 1}\right)_k$ is convergent.

From $\left(x_{n_k^1}\right)_k$. extract a subsequence: $\left(x_{n_k^2}\right)_k$. s.t.

$\left(x_{n_k^2, 1}\right)_k$ and $\left(x_{n_k^2, 2}\right)_k$ are convergent.

...

Repeat & continue inductively: $a = 1, 2, \ldots$ $\left(x_{n_k^a}\right)_{k \geq 1}$: $\left(x_{n_k^a, j}\right)_k$ convergent in $\mathbb{C}$

$\forall\, 1 \leq j \leq a$

$\dashrightarrow$ Extract the "diagonal": $\left(x_{n_k^k}\right)_{k \geq 1}$: $\forall j \geq 1$: $\left(x_{n_k^k, j}\right)_k$ is conv. in $\mathbb{C}$

Claim $\left(x_{n^k_k}\right)_k$ converges weakly.

$\Longleftrightarrow$ $\forall\, l \in H$, $\underbrace{\left(\langle l, x_{n^k_k}\rangle\right)_k}_{\text{Cauchy property.}}$ is convergent to some $\langle l, z\rangle$

Fix $\varepsilon > 0$. Let N_0 be s.t. $\left(\sum_{j=N_0}^{\infty} |l_j|^2\right)^{1/2} < \frac{\varepsilon}{4}$.

$$l = \sum_{j=1}^{\infty} l_j e_j$$

Let N be s.t. $\cancel{\left(x_{n^k_{k,j}}\right)_k}$

$\forall a, b \geq N$:

$$\forall\, 1 \leq j \leq N_0. \quad |x_{n^a_{a,j}} - x_{n^b_{b,j}}| < \frac{\varepsilon}{2 N_0 \|l\|}$$

$$\rightarrow \quad |\langle l, x_{n^a_a}\rangle - \langle l, x_{n^b_b}\rangle| \leq$$

$$\leq \underbrace{\sum_{j=1}^{N_0} |l_j| \cdot |x_{n^a_{a,j}} - x_{n^b_{b,j}}|}_{\leq \|l\| \cdot \underbrace{\sum_{j=1}^{N_0} |x_{n^a_{a,j}} - x_{n^b_{b,j}}|}_{< \varepsilon/2}} + \underbrace{\sum_{j=N_0+1}^{\infty} |l_j| \cdot |x_{n^a_{a,j}} - x_{n^b_{b,j}}|}_{\leq \underbrace{\left(\sum_{j=N_0}^{\infty} |l_j|^2\right)^{1/2}}_{< \frac{\varepsilon}{4}} \cdot \underbrace{\left(\sum |\ \ |^2\right)^{1/2}}_{\leq 2}} < \ldots$$

<u>Metrizability</u>:

$H^* \simeq H \xrightarrow{\text{separable}}$: $(\ell_n)_{n\geq 1}$ - dense in H^*

$\ell_n \xrightarrow{\text{semimetric.}}$ $d_n(x,y) = |\ell_n(x-y)|$

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \cdot \frac{d_n(x,y)}{1+d_n(x,y)}. \implies \text{metric on } H$$

induced topology is the weak topology.