L15

Spectral Decompositions of Compact Operators

Theorem [The Canonical Form of Compact Operators; The SVD Decomposition]. Assume H is a separable Hilbert space and $T: H \to H$ is a compact operator. Then there are two orthonormal sets $\{u_k\}_{k=1}^{N_0}$ and $\{v_k\}_{k=1}^{N_0}$ and positive real numbers $(s_k)_{k=1}^{N_0}$ so that if $N_0 = \infty$, $\lim_{k\to\infty} s_k = 0$, and so that:

$$T = \sum_{n=1}^{N_0} s_n \cdot \langle \cdot, u_n \rangle v_n .$$

where the series (if $N_0 = \infty$) converges in operator norm, i.e.

$$\lim_{N\to\infty} \| T - \sum_{k=1}^{N} s_k \langle \cdot, u_k \rangle v_k \| = 0 .$$

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Definition The numbers $(s_k)_{k=1}^{N_0}$ are called the singular values of T. The vectors $(u_n)_{n\geq 1}$ are called the right singular vectors, the vectors $(v_n)_{n\geq 1}$ are called the left singular vectors.

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Remark: N_0 is the rank of T, i.e. dimension of range $\mathrm{Ran}(T)$.

Theorem [The Hilbert-Schmidt Theorem] Let A be a self-adjoint operator on H. (separable Hilbert space). Then there is a complete orthonormal basis $\{u_n\}_{n\geq 1}$ for H so that $A u_n = \lambda_n u_n$, and $\lim_{n\to\infty} \lambda_n = 0$.

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<u>Theorem</u> [The Riesz-Schauder Theorem]. Let A be a compact operator on $\mathcal{H}$. Then the spectrum of A, $\sigma(A)$, is a discrete set having no limit point except perhaps $\lambda = 0$. Further, any non-zero $\lambda \in \sigma(A)$ is an eigenvalue of finite multiplicity (i.e. the corresponding space of eigenvectors is finite dimensional).

How to prove these results: (The SVD in particular).

$$T : H \to H, \quad \text{compact.}$$

$$T^*T : H \to H, \quad \text{compact} \qquad - \text{ self-adjoint } (T^*T)^* = T^*T$$

$$- \text{ positive semidefinite: } \langle T^*Tx, x\rangle \geq 0, \quad \forall x \in H.$$

--> [<u>Key Step</u>: $\exists\, x \in H$, $\|x\| = 1$ s.t. $T^*T x = \|T^*T\| \cdot x$.

--> $\lambda_1 = \|T^*T\| \in \sigma(T^*T)$: $E_1 = \ker(T^*T - \|T^*T\| \ldots$

P_1 orth. proj. onto E_1, $\{ \cancel{e_1}\, u_1, u_2, \ldots u_{r_1}\}$ ONB in E_1

--> "peel-off" λ_1:

Construct: $$T_1 = T - \sqrt{\lambda_1} \sum_{k=1}^{r_1 = \dim E_1} \langle \cdot, u_k\rangle v_k$$

where $$v_k = \frac{1}{\sqrt{\lambda_1}} T u_k$$

Then.

$$R_1 = T_1^* \cdot T_1 = T^*T - \lambda_1 P_1 \ :$$

Repeat

Lemma 1. For any $T \in B(H)$,

$$\ker(T) = \ker(T^*T).$$

Proof.

- $\ker(T) \subset \ker(T^*T)$ obvious.
- converse: let $x \in \ker(T^*T)$: $T^*Tx = 0 \Rightarrow \langle T^*Tx, x \rangle = 0$

$\rightarrow \langle Tx, Tx \rangle = 0 \Rightarrow \|Tx\|^2 = 0 \Rightarrow Tx = 0 \Rightarrow x \in \ker(T)$

$\ker(T^*T) \subset \ker(T)$. ∎

Let $E_0 = \ker(T)$. $E_0 \subset H$ is a closed subspace.

let $H_0 = E_0^\perp = \{x \in H : \forall y \in E_0, \langle x, y \rangle = 0\}$

[**Claim**: For any closed subspace $E \subset H$, $E^\perp$ is a closed subspace and $H = E \oplus E^\perp$. Furthermore there exists: $P_E : H \to H$, $\mathrm{Ran}(P_E) = E$.

$P_E^* = P_E$, $P_E^2 = P_E$

(orthogonal projection onto E).]

$H = H_0 \oplus E_0$.

$\boxed{\text{Since } T \text{ is compact}}$

$T|_{H_0} : H_0 \to H$, $T|_{H_0}$ is compact.

$T^*T|_{E_0} = 0.$

Lemma 2. Assume $A \in B(H)$ is a bounded operator on H, and $E \subset H$ is an invariant subspace for A, i.e. $AE \subset E$. Then $E^\perp$ is an invariant subspace for A^*. In particular, if $A = A^*$ then $E^\perp$ is invariant for A

Proof.

If $E \subset H$, $A(E) \subset E$: $\forall x \in E$, $Ax \in E$.

let $y \in E^\perp$. Want: $A^* y \in E^\perp$

$\forall x \in E$: $\langle A^* y, x \rangle = \langle \underset{\in E^\perp}{y}, \underbrace{Ax}_{\in E} \rangle = 0. \to A^* y \in E^\perp$.

$\blacksquare$

Consequence of this Lemma: H_0 is invariant for $T^* T$.

$$T^* T\Big|_{H_0} : H_0 \longrightarrow H_0.$$

Eventually: $\boxed{N_0 = \dim H_0}$

If. $\dim H_0 < \infty \to T$ has finite rank and $\operatorname{rank}(T) = N_0 = \dim H_0$

If $\dim H_0 = \infty \to T$ has infinite rank.
$\dashrightarrow$ infinite # of non-zero singular values (and eigenvalues).

Lemma 3. Assume $T \in B(H)$ is a bounded operator (not necessary compact)

Then:

(a) $\|T^*\| = \|T\|$

(b) $\|T^* T\| = \|T\|^2$

Proof.

(a) $\|T^*\| = \sup\limits_{\|x\|=1} \|T^* x\| = \sup\limits_{\|x\|=1} \sup\limits_{\|y\|=1} |\langle T^* x, y \rangle| = \sup\limits_{\|y\|=1} \sup\limits_{\|x\|=1} |\langle x, Ty \rangle| =$

$= \sup\limits_{\|y\|=1} \|Ty\| = \|T\|$.

(b)

$$\|T^* T\| \leq \|T^*\| \cdot \|T\| = \|T\| \cdot \|T\| = \|T\|^2$$

$$\|T^* T\| = \sup_{\|x\|=1} \sup_{\|y\|=1} |\langle T^* T x, y\rangle| \geq \sup_{\|x\|=1} |\langle T^* T x, x\rangle| =$$

$$= \sup_{\|x\|=1} \|Tx\|^2 = \|T\|^2.$$

Lemma 4 [Key Step]. Assume $T \in B(H)$ is compact. There exists $x \in H$, $\|x\| = 1$ s.t. $\|Tx\| = \|T\|$ and $T^* T x = \|T\|^2 x$.

Proof.

1. $\|T\| = \sup_{\|y\|=1} \|Ty\| \dashrightarrow \exists (y_n)_n : \|y_n\| = 1$ s.t. $\lim_{n\to\infty} \|Ty_n\| = \|T\|$

$(y_n)_n \subset U_1 = \{y \in H : \|y\| \leq 1\}$ closed unit ball.

But: U_1 is weakly compact $\Rightarrow \exists (y_{n_k})_k$ convergent weakly.

let $x = \text{weak-}\lim_{k\to\infty} y_{n_k}$: $y_{n_k} \xrightarrow{w} x, \in U_1$.

2. But T is compact: $(Ty_{n_k})_k$ converges strongly (in H-norm):

$$\lim_{k\to\infty} Ty_{n_k} = Tx, \text{ in } \|\cdot\| \text{ topology}.$$

Therefore: $\|Tx\| = \lim_{k\to\infty} \|Ty_{n_k}\| = \|T\|$:

3. Claim: $\|x\|=1$. $\qquad x \in U_1: \|x\| \leq 1$.

If $\|x\|<1 \longrightarrow \left\| T\frac{x}{\|x\|}\right\| = \frac{1}{\|x\|}\cdot \|Tx\| = \frac{\|T\|}{\|x\|} > \|T\|$.

contradiction

Therefore: $\exists x \in H$, $\|x\|=1$ s.t. $\|Tx\| = \|T\|$.

4.

$$0 \leq \left\| T^*Tx - \|T\|^2 x \right\|^2 = \|T^*Tx\|^2 - 2\cdot\|T\|^2\cdot\underbrace{\langle T^*Tx, x\rangle}_{\|Tx\|^2=\|T\|^2} + \|T\|^4$$

$$= \|T^*Tx\|^2 - \|T\|^4 \leq \|T^*T\|^2 - \|T\|^4 = \|T\|^4 - \|T\|^4 = \underline{\underline{0}}.$$

$$\longrightarrow \underline{T^*Tx = \|T\|^2 x}. \qquad \boxed{}$$

Let $\lambda_1 = \|T\|^2 = \|T^*T\|$.

let $E_1 = \ker(T^*T - \lambda_1 \cdot 1) \neq \{0\}$.

let $r_1 = \dim E_1 < \infty$ (because T^*T is compact).

let P_1 be the orthogonal projection onto E_1.

let $\{u_1, u_2 \dots u_{r_1}\}$ be ONB in E_1. $\longrightarrow \underline{\sum_{k=1}^{r_1} \langle \cdot, u_k\rangle u_k = P_1}$.

Set: $V_1 = \frac{1}{\sqrt{\lambda_1}} Tu_1$, $V_2 = \frac{1}{\sqrt{\lambda_1}} Tu_2, \dots \; V_{r_1} = \frac{1}{\sqrt{\lambda_1}} Tu_{r_1}$

Claim: $\{v_1 \dots v_{r_1}\}$ ONB for $T(E_1)$.

Pf: $\langle v_k, v_j\rangle = \frac{1}{\lambda_1}\langle Tu_k, Tu_j\rangle = \frac{1}{\lambda_1}\langle \underbrace{T^*Tu_k}_{\lambda_1 u_k}, u_j\rangle = \langle u_k, u_j\rangle = \delta_{kj}$.

$\longrightarrow$ ON set