(L16)

Compact Operators (2)

$\{u_1 \ldots u_{r_1}\}$ ONB for E_1 : E_1 eigenspace for T^*T associated to eigenvalue λ_1

$$v_1 \ldots v_{r_1} : \left\{ \frac{Tu_1}{\|Tu_1\|}, \frac{Tu_2}{\|Tu_2\|}, \ldots \right\}.$$

$\|Tu_k\| = \sqrt{\lambda_1} = s_1$ (first singular value).

↳ because: $(T^*T)u_k = \lambda_1 u_k \to \langle T^*T u_k, u_k\rangle = \lambda_1$, $\|Tu_k\|^2 = \lambda_1$

Claim:

$$\langle v_i, v_j\rangle = \delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i\neq j \end{cases}.$$

$$\left\langle \frac{Tu_i}{\|Tu_i\|}, \frac{Tu_j}{\|Tu_j\|} \right\rangle = \frac{1}{\underbrace{\|Tu_i\|\cdot\|Tu_j\|}_{\lambda_1}} \langle \underbrace{T^*T u_i}_{\lambda_1 u_i}, u_j\rangle = \langle u_i, u_j\rangle.$$

Step 2. (after "key step"):

Set:
$$T_1 = T - \left(s_1\langle\cdot, u_1\rangle v_1 + s_1\langle\cdot, u_2\rangle v_2 + \cdots + s_1\langle\cdot, u_{r_1}\rangle v_{r_1}\right)$$
$$= T - s_1 \cdot \sum_{k=1}^{r_1} \langle\cdot, u_k\rangle v_k.$$

where: $s_1 = \sqrt{\lambda_1} = \sqrt{\|T^*T\|} = \|T\|$

$$T_1^* = T^* - s_1 \sum_{k=1}^{r_1} \left(\langle\cdot, u_k\rangle v_k\right)^*$$

Claim: $\big(\underbrace{\langle\cdot, u_k\rangle v_k}_{R}\big)^* = \langle\cdot, v_k\rangle u_k.$

Pf: By def: $\langle R^* x, y\rangle = \langle x, Ry\rangle = \langle x, \langle y, u_k\rangle v_k\rangle = \langle u_k, y\rangle \cdot \langle x, v_k\rangle = \langle\langle x, v_k\rangle u_k, y$

$$T_1^* = T^* - s_1 \sum_{k=1}^{r_1} \langle \cdot, v_k \rangle u_k$$

$$T_1^* T_1 = \Big(T^* - s_1 \sum_{k=1}^{r_1} \underbrace{\langle \cdot, v_k \rangle u_k}\Big)\Big(T - s_1 \sum_{j=1}^{r_1} \underbrace{\langle \cdot, u_j \rangle v_j}\Big) =$$

$$= T^* T - s_1 \sum_{k=1}^{r_1} \langle T\cdot, v_k \rangle u_k - s_1 \sum_{j=1}^{r_1} \langle \cdot, u_j \rangle T^* v_j +$$

$$+ s_1^2 \sum_{k,j=1}^{r_1} \langle \cdot, u_j \rangle \underbrace{\langle v_j, v_k \rangle}_{\delta_{jk}} u_k =$$

$$v_k = \frac{T u_k}{\|T u_k\|} \dashrightarrow T^* v_k = \frac{T^* T u_k}{\|T u_k\|} = \frac{\lambda_1 u_k}{\sqrt{\lambda_1}} = \underbrace{\sqrt{\lambda_1}}_{s_1} u_k.$$

$$= T^* T - s_1^2 \underbrace{\sum_{k=1}^{r_1} \langle \cdot, u_k \rangle u_k}_{P_1} - s_1^2 \underbrace{\sum_{j=1}^{r_1} \langle \cdot, u_j \rangle u_j}_{P_1} + s_1^2 \underbrace{\sum_{k=1}^{r_1} \langle \cdot, u_k \rangle u_k}_{P_1} =$$

$$= T^* T - \lambda_1 \cdot P_1$$

Summarize:

$$\begin{array}{ccl} T & \longrightarrow & T_1 = T - s_1 \sum_{k=1}^{r_1} \langle \cdot, u_k \rangle v_k \\ \downarrow & & \downarrow \\ T^* T & \dashrightarrow & T_1^* T_1 = T^* T - \lambda_1 P_1, \quad P_1 = \underbrace{\sum_{k=1}^{r_1} \langle \cdot, u_k \rangle u_k}_{\text{orthogonal projection onto } E_1} \end{array}$$

Lemma 5. $\|T_1^* T_1\| < \lambda_1 = \|T^* T\|$.

Furthermore, if $\lambda_2 = \|T_1^* T_1\|$, $\lambda_2 \in \sigma(T^* T)$.

Proof.

Let $\lambda_2 = \|T_1^* T_1\|$.

By same argument as before, $\exists x \in H, \|x\| = 1$ s.t. $T_1^* T_1 x = \lambda_2 \cdot x$.

$\dashrightarrow \lambda_2 \in \sigma(T_1^* T_1)$.

$\lambda_2 = \langle T_1^* T_1 x, x\rangle = \|T_1 x\|^2 \underline{= \|T_1^* T_1\|}$

—.

Let $H_0 = E_1 \oplus H_1$, $H_1 = E_1^\perp \cap H_0$ (orthogonal complement of E_1 in H_0)

$x = x_E + x_H$, where $x_E \in E_1$,
$x_H \in H_1$.

$$T_1^* T_1 x_E = T^* T \underbrace{x_E}_{\in E_1} - \lambda_1 \cdot P_{E_1} x_E = \lambda_1 \cdot x_E - \lambda_1 \cdot x_E = 0.$$

$$T_1^* T_1 x = \underbrace{T_1^* T_1 x_H}_{} = T^* T x_H,$$

E_1 invariant to T^*T
$\Downarrow$
H_1 invariant to T^*T.

$$T^* T x_H \in H_1.$$

$$\lambda_2 \cdot x = T^* T x_H \in H_1 \dashrightarrow . x_E = 0.$$

$x = x_H \in H_1$; $\|x_H\| = 1$.

$$\lambda_2 = \|T_1^* T_1 x\| \overset{\|T^*Tx\|}{=} \|T_1 x\|^2 = \|Tx\|^2.$$

If $\lambda_2 = \lambda_1 \Rightarrow \|Tx\|^2 = \lambda_1 \Rightarrow x \in E_1 \longrightarrow$ contradiction.

Therefore: $\underline{\lambda_2 < \lambda_1}$.

and: $T^* T x = T_1^* T_1 x = \lambda_2 \cdot x \Rightarrow \lambda_2 \in \sigma(T^* T)$.

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Next Proceed inductively:

$$T \longrightarrow T_1 = T - \sqrt{\lambda_1} \sum_{k=1}^{r_1} \langle \cdot, u_k \rangle v_k \longrightarrow T_2 = T_1 - \sqrt{\lambda_2} \sum_{k=r_1+1}^{r_1+r_2} \langle \cdot, u_k \rangle v_k \longrightarrow \cdots$$

$$\downarrow$$

$$T^*T \dashrightarrow T_1^* T_1 = T^*T - \lambda_1 P_1. \longrightarrow T_2^* T_2 = T_1^* T_1 - \lambda_2 P_2. \rightarrow \cdots$$

$P_1; E_1, \lambda_1$

$\{u_1, \dots, u_{r_1}\}$ ONB

$$v_k = \frac{Tu_k}{\|Tu_k\|} = \frac{1}{\sqrt{\lambda_1}} T u_k$$

$r_1 = \dim E_1$

$P_2; E_2, \lambda_2$

$\{u_{r_1+1}, \dots, u_{r_1+r_2}\}$.

$$v_k = \frac{Tu_k}{\|Tu_k\|} = \frac{1}{\sqrt{\lambda_2}} T u_k$$

$r_2 = \dim E_2$

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we obtain:

$$T_n = T - \sum_{\ell=1}^{n} \sqrt{\lambda_\ell} \sum_{k \in I_\ell} \langle \cdot, u_k \rangle v_k.$$

where: $I_\ell = \{ r_1 + \dots + r_{\ell-1} + 1, \dots, r_1 + r_2 + \dots + r_\ell \}$.

$$T_n^* T_n = T^*T - \sum_{\ell=1}^{n} \lambda_\ell \cdot P_\ell.$$

and $\lambda_1 > \lambda_2 > \dots > \lambda_n$

If. $\exists \tilde{N}_0$ s.t. $\lambda_{\tilde{N}_0+1} = 0, \lambda_{N_0} > 0 \rightarrow$ then $T^*_{\tilde{N}_0+1} \cdot T_{\tilde{N}_0+1} = 0 \Rightarrow$

$$\Rightarrow T^*T = \sum_{n=1}^{\tilde{N}_0} \lambda_n P_n \longrightarrow T = \sum_{k=1}^{N_0} \sqrt{\lambda_k} \cdot \langle \cdot, u_k \rangle v_k.$$

The alternative: $(\lambda_n)_{n\geq 1}$ is strictly decreasing but $\forall n$: $\lambda_n > 0$.

Consequence of Lemma 5:

$$\| T_n^* T_n \| = \| T^* T - \sum_{\ell=1}^{n} \lambda_\ell \cdot P_\ell \| = \lambda_{n+1} < \lambda_n .$$

Lemma 6: $\forall k \neq j$, $P_k \cdot P_j = 0$.

Proof:

Let $\lambda_k \neq \lambda_j$ denote the corresponding eigenvalues of $T^* T$.

Let $x \in E_k = \mathrm{Ran}(P_k)$, $y \in E_j = \mathrm{Ran}(P_j)$.

$$\lambda_k \cdot \langle x, y \rangle = \langle \lambda_k x, y \rangle = \langle T^* T x, y \rangle = \langle x, T^* T y \rangle =$$

$$= \langle x, \lambda_j y \rangle = \lambda_j \langle x, y \rangle \Rightarrow \langle x, y \rangle = 0.$$

$$\Rightarrow E_k \perp E_j \rightarrow P_k \cdot P_j = 0.$$

Let $F = \mathrm{closure}(E_1 \oplus E_2 \oplus \cdots \oplus E_n \oplus \cdots)$.

Lemma 7: $F = H_0 = E_0^{\perp} = (\ker(T))^{\perp} = (\ker(T^* T))^{\perp}$.

Therefore, $\lim_{n\to\infty} \lambda_n = 0$.

Proof.

If. $F = H_0 \longrightarrow$ ok.

Assume $F \subset H_0$, $F \neq H_0$., proper subspace.

~~Tesdere~~ $T^*T\big|_{F}$ is compact., $F \subset H_0$ closed subspace.

If. $\lim\limits_{n\to\infty} \lambda_n = \mu > 0. \rightarrow \| T^*T\big|_F x \|^2 = ?$

$$\underbrace{x = \sum_{k=1}^{\infty} x_k}^{x \in F}, \quad x_k \in E_k.$$

where: $x_k = P_k x.$

$$\| T^*T(\sum_{k=1}^{\infty} x_k) \|^2 = \| \sum_{k=1}^{\infty} \underbrace{T^*T x_k}_{\lambda_k x_k.} \|^2 = \| \sum_{k=1}^{\infty} \lambda_k x_k \|^2 =$$

$$= \langle \sum_{k=1}^{\infty} \lambda_k x_k, \sum_{j=1}^{\infty} \lambda_j x_j \rangle = \sum_{k,j\geq 1} \lambda_k \lambda_j \underbrace{\langle x_k, x_j \rangle}_{\delta_{kj} \cdot \|x_k\|^2} =$$

$$= \sum_{k=1}^{\infty} \lambda_k^2 \cdot \|x_k\|^2 \geq \mu^2 \sum_{k=1}^{\infty} \|x_k\|^2 = \mu^2 \|x\|^2.$$

Thus: $\| T^*T x \| \geq \mu \cdot \|x\|$, for $x \in F$.

Furthermore: $$T^*T x = \sum_{k=1}^{\infty} \lambda_k x_k$$

Therefore: $T^*T : F \rightarrow F$ is invertible:

<u>Because</u>:

① Take $y \in F$: let $y = \sum_{k=1}^{\infty} y_k \longrightarrow$ let $x = \underbrace{\sum_{k=1}^{\infty} \frac{1}{\lambda_k} y_k}_{\text{convergent in } F.}$

$$\|x\|^2 = \sum_{k=1}^{\infty} \frac{1}{\lambda_k^2} \|y_k\|^2 \leq \frac{1}{\mu^2} \|y\|^2$$

$\Rightarrow T^*Tx = y \longrightarrow T^*T$ is surjective.

② If $T^*Tx = 0$: $\quad 0 = \|T^*Tx\| \geq \mu\|x\| \Rightarrow x = 0.$

$\longrightarrow T^*T$ is injective.

$T^*T : F \to F$ invertible.

$$\left.\begin{array}{l}\underbrace{(T^*T|_F)^{-1} \cdot \underbrace{(T^*T)|_F}_{\text{compact.}}}_{\text{compact.}} = 1_F\end{array}\right\} \longrightarrow \underline{\underline{\dim F < \infty}}.$$

Contradiction.

Contradiction: From $\cancel{\lim_{n\to\infty} \lambda_n = \mu > 0}$.

Thus: $\lim_{n\to\infty} \lambda_n = 0$. $\Rightarrow \lim_{n\to\infty} \|T_n^*T_n\| = 0.$

$\longrightarrow$ $F = H_0$, there is no space "left" between F and H_0.

<u>Conclusion</u>: $T^*T = \sum_{k=1}^{\infty} \lambda_k P_k$, convergence in operator norm.

and $T = \sum_{k=1}^{\infty} \sqrt{\lambda_k}\langle\cdot, u_k\rangle v_k$, convergence in operator norm.

(SVD)