L17

Compact Operators (3)

<u>Theorem</u> [The Hilbert-Schmidt Theorem] Let $T = T^*$ be a compact self-adjoint operator on a separable Hilbert space H. Then there is a complete orthonormal basis $\{u_n\}_{n\geq 1}$ for H so that

$Au_n = \lambda_n u_n$, and $\lim_{n\to\infty} \lambda_n = 0$.

<u>Proof</u>

At this point we know T admits the SVD decomposition:

$$T \to T^*T = T^2 = \sum_{n=1}^{\infty} \mu_n \cdot P_n, \quad \text{where } \sigma(T^2) = \{\mu_1, \mu_2, \dots\} \subset \mathbb{R}$$

$\mu_1 > \mu_2 > \dots \geq 0$.

each P_n is an orthogonal projection of finite rank.

$$\to T = \sum_{k=1}^{\infty} s_k \langle \cdot, u_k \rangle v_k, \quad \text{where } s_k^2 \in \sigma(T^2).$$

1) $s_k^2 = \mu_n$

2) $\{u_1, u_2, \dots\}$, $\{v_1, v_2, \dots\}$ ONB set

$\{u_k : s_k^2 = \mu_n\}$ ONB for $E_n = \mathrm{Ran}(P_n)$.

Let P_0 denote the orthogonal projection onto $\ker(T) = \ker(T^2) =: E_0$.

<u>Note</u>: last time, Lemma 6 showed: $P_k \cdot P_j = 0$, $k \neq j$, $\forall k, j \geq 0$

<u>Claim:</u> $T\cdot P_n = P_n \cdot T$ so that $T(E_n) \subset E_n$, $\forall n \geq 0$.

Show the claim:

We know: $T^2 P_n = P_n \cdot T^2 = \mu_n \cdot P_n$ (P_n are spectral projections for T^2).

Let $x \in E_n = \mathrm{Ran}(P_n)$.

$$T^2 x = \mu_n \cdot x \implies \underbrace{T \cdot T^2}_{T^2 \cdot T} x = \mu_n T x$$

$$T^2 \cdot Tx = \mu_n Tx \Rightarrow Tx \in E_n.$$

$$\Rightarrow T(E_n) \subset E_n.$$

Take $x \in H$. $x = \sum_{m=1}^{\infty} \underbrace{P_m x}_{\in E_m} + \underbrace{P_0 \cdot x}_{\in \ker(T)}$, convergent in H-norm.

Fix $n \geq 1$.

$$P_n x = \sum_{m=1}^{\infty} \underbrace{P_n P_m}_{\delta_{n,m} \cdot P_n} x + P_n P_0 x = P_n x$$

$$Tx = \sum_{m=1}^{\infty} \underbrace{T P_m}_{\lambda_m P_m} x + \underbrace{T P_0}_{0} x = \sum_{m=1}^{\infty} \lambda_m P_m x$$

$$\left.\begin{aligned} P_n T x &= \sum_{m=1}^{\infty} \lambda_m \underbrace{P_n P_m}_{\delta_{n,m} \cdot P_n} x = \lambda_n P_n x \\ T \underbrace{P_n x}_{\in E_n} &= \lambda_n P_n x \end{aligned}\right\} \Rightarrow \underline{\underline{P_n T = T P_n.}}$$

Thus: $T\big|_{E_n} : E_n \to E_n$.

(Subclaim): $T\big|_{E_n} = \left(T\big|_{E_n}\right)^*$ is also self-adjoint.

...

let $A = T\big|_{E_n}$, $A : E_n \to E_n$., $\dim E_n = r_n < \infty$

$A^* = A$.

$A^2 = \mu_n \cdot 1_{E_n}$.

$\dashrightarrow \exists\, \tilde{U} = [\tilde{u}_1 | \dots | \tilde{u}_{r_n}] : \tilde{u}^* \cdot \tilde{u} = 1_{E_n}$.

$\{\tilde{u}_1, \dots \tilde{u}_{r_n}\}$ ONB for E_n.

$\Rightarrow A\tilde{u}_k = \pm\sqrt{\mu_n}\, \tilde{\mu}_k$, $1 \le k \le r_n$.

let $m_n^+ = |\{k : A\tilde{u}_k = \sqrt{\mu_n}\, \tilde{u}_k\}| \to$ multiplicity of $\sqrt{\mu_n}$

$m_n^- = |\{k : A\tilde{u}_k = -\sqrt{\mu_n}\, \tilde{u}_k\}| \to$ multiplicity of $-\sqrt{\mu_n}$ for A.

$\Rightarrow$ $T\tilde{u}_k = \sqrt{\mu_n}\, \tilde{u}_k$, if $1 \le k \le m_n^+$.

$T\tilde{u}_k = -\sqrt{\mu_n}\, \tilde{u}_k$, if $m_n^+ + 1 \le k \le r_n = m_n^+ + m_n^-$.

Put everything together: $\{u_1, u_2, \dots\} \equiv \bigcup_{n \ge 0} \bigcup_{k=1}^{r_n} \{\tilde{U}_k\}$

ONB for H.

$T u_k = \pm\sqrt{\mu_n}\, u_k$,

In particular: $S_k = |\lambda_k|$

Let $\lambda_k = \langle T u_k, u_k \rangle \longrightarrow T u_k = \lambda_k u_k$:

By SVD theorem: $\lim \lambda_n = \lim \mu_n = 0$

Schatten Classes of Compact Operators

$$\mathcal{J}_p = \{ T: H \to H,\ T \text{ compact s.t. } \underbrace{(s_1, s_2, \dots)}_{\text{sequence of singular values}} \in \ell^p \}.$$

For $p=\infty \dashrightarrow \sup_{k\geq 1} s_k < \infty$ always true: $\underline{\mathcal{J}_\infty = K(H)}$.

For $0<p<\infty \dashrightarrow \sum_{k\geq 1} |s_k|^p < \infty$.

For $p=0 \dashrightarrow \operatorname{rank}(T) < \infty$. : $\mathcal{J}_0$ = set of finite rank operators.

<u>Theorem</u>. For $1 \leq p \leq \infty$, $T \in \mathcal{J}_p$, $\|T\|_p = \|(s_1, s_2, \dots)\|_p$ defines a norm on $\mathcal{J}_p$. $(\mathcal{J}_p, \|\cdot\|_p)$ is a Banach space.

<u>Sketch of proof</u>.

$p=1$: $\|T+S\|_1 \leq \|T\|_1 + \|S\|_1$, $T, S \in \mathcal{J}_1$.

$$\sum_{k\geq 1}^{\infty} s_k(T+S) \leq \sum_{k\geq 1}^{\infty} s_k(T) + \sum_{k\geq 1}^{\infty} s_k(S)$$

$$s_1(T+S) = \|T+S\| \leq \|T\| + \|S\| = s_1(T) + s_1(S).$$

NOTE: $\boxed{s_k(T+S) \text{ may not be } \leq s_k(T) + s_k(S), \text{ when } k\geq 2.}$

However:

$$\sum_{k=1}^{N} s_k(T+S) \le \sum_{k=1}^{N} s_k(T) + \sum_{k=1}^{N} s_k(S)$$

for any $N \ge 1$.

Why:

$$\underbrace{\sum_{k=1}^{N} s_k(T+S)}_{\displaystyle \max_{\substack{\{u_1\ldots u_N\}\ O.N. \\ \{v_1\ldots v_N\}\ O.N.}} \left[\mathrm{Re} \sum_{k=1}^{N} \langle (T+S) u_k, v_k \rangle \right]}$$

For $p \ge 1$. ⇢ see: Ky Fan Inequalities.

For each $p \ge 1$: $\mathcal{J}_0$ is dense in $\mathcal{J}_p$ w.r.t. $\|\cdot\|_p$ norm.

Two special classes:

$\boxed{p=2}$ $\quad \mathcal{J}_2 = \left\{ T : H \to H,\ T \text{ compact s.t. } \sum_{k=1}^{\infty} (s_k)^2 < \infty \right\}$

↳ Class of Hilbert-Schmidt operators; $\|T\|_2 \equiv$ Frobenius norm.

Theorem 1: If $\{u_1, u_2, \ldots\}$ is ONB for H,

$$\|T\|_2^2 = \sum_{k\ge 1} \|T u_k\|^2 = \sum_{k=1}^{\infty} \langle T^* T u_k, u_k \rangle.$$

Proof: check.

$\boxed{p=1}$ $\quad j_1 = \{ T: H \to H,\ T \text{ compact}: \sum_{k\geq 1} s_k < \infty \}$.

↳ ~~Class~~ Set of trace class operators.

$\|T\|_1 = \sum_{k\geq 1} s_k(T)$: the nuclear norm.

Theorem 2 If $\{u_1, u_2, \ldots\}$, $\{v_1, v_2, \ldots\}$ ONB in H then:

$$\sum_{k=1}^{\infty} \langle T u_k, u_k \rangle = \sum_{k=1}^{\infty} \langle T v_k, v_k \rangle .$$

Notation: Trace : $j_1 \longrightarrow \mathbb{R}$, $\quad tr(T) = \sum_{k=1}^{\infty} \langle T u_k, u_k \rangle$

where $\{u_1, u_2, \ldots\}$ ONB in H.

How to show the Theorem 2:

1) show first on j_0 : finite rank. ⟷ On matrices $A \in \mathbb{C}^{n \times n}$.

$$tr(A) = \sum_{k=1}^{n} \langle A e_k, e_k \rangle$$

2). $\left| \sum_{k=1}^{\infty} \langle T u_k, u_k \rangle \right| \leq \sum_{k=1}^{\infty} s_k = \|T\|_1$

T is Hilbert-Schmidt $\Longleftrightarrow$ $T^* T$ is trace class

$T \in j_2 \qquad T^* T \in j_1$

and: $\|T\|_2^2 = tr(T^* T)$

FUNCTIONAL CALCULUS

Goal: 1). Banach algebras

2) Spectrum w.r.t a Banach algebra

3). Gelfand's Formula: $r_A(T) = \lim_{n\to\infty} \|T^n\|_A^{1/n}$.

4). Resolvent map: $R_T(z) = (z\cdot 1 - T)^{-1}$.

5) Holomorphic Functional Calculus.

Definition. A Banach algebra $\mathcal{U}$ is a complex Banach space together with a product operation $\cdot$, such that: $\forall x, y, z \in \mathcal{U}$:

(1) Distributive: $(x+y)\cdot z = x\cdot z + y\cdot z$, $x\cdot(y+z) = x\cdot y + x\cdot z$

(2) Associative: $(x\cdot y)\cdot z = x\cdot(y\cdot z)$.

(3) Submultiplicative: $\|x\cdot y\| \leq \|x\|\cdot\|y\|$.

Typical Example: $\mathcal{U} \subset B(X) = \{T : X \to X,\ T \text{ bounded}\}$.

X: Banach space.

$\mathcal{U}$ algebra (closed w.r.t. multiplication in $B(X)$), complete w.r.t. $\|\cdot\|_{B(X)}$

then, $\mathcal{U}$ is a Banach algebra

Definition A Banach algebra $\mathcal{U}$ is said to have **identity** if there exists $e \in \mathcal{U}$, $\|e\| = 1$ s.t. $e \cdot x = x \cdot e = x$, $\forall x \in \mathcal{U}$.

Fact. When $\mathcal{U}$ does not have identity, one can "add" an identity:

$$\mathcal{U} \longrightarrow \tilde{\mathcal{U}} = \mathcal{U} \oplus \mathbb{C}$$

with:

Algebra
$$\begin{cases} (x_1, a_1) + (x_2, a_2) = (x_1 + x_2, a_1 + a_2). \\ (x_1, a_1) \cdot (x_2, a_2) = (x_1 \cdot x_2 + a_1 x_2 + a_2 x_1, a_1 \cdot a_2) \end{cases}$$

Norm. $- \; \|(x, a)\|_{\tilde{\mathcal{U}}} = \|x\|_{\mathcal{U}} + |a|.$

Then $(0, 1) \in \tilde{\mathcal{U}}$ is the identity element.

Theorem 1: Assume $T \in \mathcal{J}_2$ $\quad \|T\|_2^2 = \sum_{k\geq 1} \|Tu_k\|^2$, $\forall \{u_1, u_2, \dots\}$ ONB.

Proof

~~Fix ε.~~

Let $T_0 = \underbrace{\sum_{k\geq 1}^{\infty} s_k \langle \cdot, u_k \rangle v_k}_{\cancel{\text{finite rank}}}$ $\cancel{\in \mathcal{J}_2 \text{ s.t. } \|T - T_0\|_2^2 = \sum_{k\geq N} s_k^2 < \varepsilon}$ be the SVD decomposition.

Let $\{e_1, e_2, \dots\}$ ONB:

$$\|T_0 e_j\|^2 = \Big\| \sum_{k\geq 1}^{\infty} s_k \langle e_j, u_k \rangle \underset{\uparrow \ \text{ONB}}{v_k} \Big\|^2 = \sum_{k\geq 1}^{\infty} s_k^2 |\langle e_j, u_k \rangle|^2$$

$$\sum_{j\geq 1} \|Te_j\|^2 = \sum_{j\geq 1} \sum_{k\geq 1} s_k^2 |\langle e_j, u_k \rangle|^2 = \sum_{k\geq 1} s_k^2 \underbrace{\sum_{j\geq 1} |\langle e_j, u_k \rangle|^2}_{\|u_k\|^2 = 1} = \sum_{k\geq 1} s_k^2$$

Theorem 2 Assume $T \in \mathcal{J}_1$

Then for any two ONB's $\{e_1, e_2, \dots\}$, $\{f_1, f_2, \dots\}$ of H,

$$\sum_{k\geq 1} \langle Te_k, e_k \rangle = \sum_{j\geq 1} \langle Tf_j, f_j \rangle.$$

Proof.

$T = \sum_k s_k \langle \cdot, u_k \rangle v_k$, uniform convergence.

We know $\sum_{k\geq 1} s_k < \infty$

$$\langle Te_j, e_j \rangle = \sum_k s_k \underbrace{\langle e_j, u_k \rangle \langle v_k, e_j \rangle}_{\text{ine' bounded.}}$$

Absolutely convergent series.

$$\sum_{j\geq 1} \sum_{k\geq 1} s_k \langle e_j, u_k \rangle \langle v_k, e_j \rangle. = ?$$

Claim:

$$a_{kj} = s_k \langle e_j, u_k\rangle\langle v_k, e_j\rangle \in \ell^1(\mathbb{N}\times\mathbb{N})$$

$$\sum_{k,j\geq 1} |a_{kj}| = \sum_{k\geq 1} s_k \sum_{j\geq 1} |\langle e_j, u_k\rangle\langle v_k, e_j\rangle| \leq$$

$$\leq \sum_k s_k \cdot \Big[\underbrace{\Big(\sum_{j\geq 1} |\langle e_j, u_k\rangle|^2\Big)^{1/2}}_{=1} \underbrace{\Big(\sum_{j\geq 1} |\langle v_k, e_j\rangle|^2\Big)^{1/2}}_{=1}\Big] = \sum_{k\geq 1} s_k < \infty.$$

Thus Fubini's theorem applies:

$$\sum_{j\geq 1}\sum_{k\geq 1} a_{kj} = \sum_{k\geq 1}\sum_{j\geq 1} a_{kj} = \sum_{k\geq 1} s_k \Big(\sum_{j\geq 1} \langle e_j, u_k\rangle\langle v_k, e_j\rangle\Big)$$

But $\langle v_k, u_k\rangle = \sum_{j\geq 1} \langle v_k, e_j\rangle\langle e_j, u_k\rangle$, because $(e_1, \dots)$ ONB.

Thus:

$$\sum_{j\geq 1} \langle Te_j, e_j\rangle = \sum_{k\geq 1} s_k \langle v_k, u_k\rangle$$

$\longrightarrow$ Conclusion

Similarly:

$$\sum_{j\geq 1} \langle Tf_j, f_j\rangle = \sum_{k\geq 1} s_k \langle v_k, u_k\rangle \quad \blacksquare$$