Gelfand Formula

Theorem [Gelfand Formula] Let $\mathcal{U}$ be a Banach algebra with identity and $x \in \mathcal{U}$. Then:

(1) $\lim\limits_{n\to\infty} \|x^n\|^{1/n}$. exists and is finite.

(2). $\max\limits_{\lambda \in \sigma_{\mathcal{U}}(x)} |\lambda|. = \lim\limits_{n\to\infty}. \|x^n\|^{1/n}.$

Definition. The spectral radius of x with respect to $\mathcal{U}$ is the smallest radius of any closed disc centered at 0 in $\mathbb{C}$ that includes $\sigma_{\mathcal{U}}(x)$. (the spectrum of x w.r.t. $\mathcal{U}$).

Makes sense: $\sigma_{\mathcal{U}}(x)$ is a non-empty compact subset of $\mathbb{C}$.

Notation: $r(x) = r_{\mathcal{U}}(x)$ denotes the spectral radius.

Remark. Consider the case when $\mathcal{U} \subset B(H)$, the algebra of bounded operators on a Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Two concepts:

1) Numerical Range: For $T \in \mathcal{U}$, $T: H \to H$,

Numerical range of T: $\{\langle Tf, f\rangle,\ f \in H,\ \|f\| = 1\} \subset \mathbb{C}$.

2) Numerical Radius:

$$rn(T) = \max_{f \in H,\ \|f\|=1} |\langle Tf, f\rangle|.$$

Recall:

1). $x : D \to \mathcal{U}$, where $D \subset \mathbb{C}$ open connected. and $\mathcal{U}$ a Banach space

is said. strongly analytic. at $z_0 \in D$ if:

$$\lim_{h \to 0} \frac{1}{h}\left(x(z_0+h) - x(z_0)\right). \quad \text{, w.r.t. } \mathcal{U}\text{-norm}$$

exists in $\mathcal{U}$.

2) $x : D \to \mathcal{U}$, $D \subset \mathbb{C}$ open connected and $\mathcal{U}$ a Banach space

is said weakly analytic. at $z_0 \in D$ if:

$\forall \ell \in \mathcal{U}'$ (dual of $\mathcal{U}$), $\displaystyle\lim_{h \to 0.} \frac{1}{h}\left(\ell(x(z_0+h)) - \ell(x(z_0))\right)$

exists in $\mathbb{C}$.

[② weakly analytic: $\underline{X} : D \to \mathbb{C}$, $\underline{X}(z) = \ell(x(z))$. is analytic.]
$\forall \ell \in \mathcal{U}'$,

Theorem. Every weakly analytic function $x : D \to \mathcal{U}$ is strongly analytic, and vice-versa.

Proof.

"$\Leftarrow$": If $x : D \to \mathcal{U}$ is strongly analytic.

$\Rightarrow$ Take $\ell : \mathcal{U} \to \mathbb{C}$, bounded (hence continuous) linear functional:

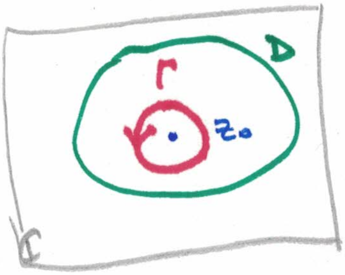
$$\underbrace{\lim_{h \to 0} \frac{1}{h}\left(\ell(x(z_0+h)) - \ell(x(z_0))\right) = \ell\Big(\underbrace{\lim_{h \to 0} \frac{1}{h}\left(x(z_0+h) - x(z_0)\right)}_{\text{exists.}}\Big)}_{\text{exists in } \mathbb{C}}$$

"$\Longrightarrow$"

Assume. $x : D \to \mathcal{U}$ is weakly analytic.

Take $\ell \in \mathcal{U}'$.

$$\ell\left(\frac{x(z_0+h)-x(z_0)}{h}\right) - \left(\frac{d}{dz}\left(\ell(x(z))\right)\right)\Bigg|_{z=z_0} = \ ?$$



Let $r>0$, $\Gamma = S_r(z_0) = \{ z \in \mathbb{C} : |z-z_0| = r \} \subset D$

Recall: $F : D \to \mathbb{C}$ an analytic (holomorphic) function: $\forall\, w \in \operatorname{int}(\Gamma) = \{ z \in \mathbb{C} : |z-z_0| < r \} = B_r(z_0)$

$$F(w) = \frac{1}{2\pi i} \oint_\Gamma \frac{F(z)}{z-w}\, dz.$$

Thus: Take $h \in \mathbb{C}$, $|h| < r$.

$\ell(x(z))$.

$$\ell\left(\frac{x(z_0+h)-x(z_0)}{h}\right) - \frac{d}{dz}\left(\ell(x(z))\right)\Bigg|_{z=z_0} =$$

$$= \frac{1}{2\pi i} \oint_\Gamma \frac{1}{h}\left(\frac{1}{z-(z_0+h)} - \frac{1}{z-z_0}\right) \ell(x(z))\, dz - \frac{1}{2\pi i} \oint_\Gamma \frac{1}{(z-z_0)^2}\, \ell(x(z))\, dz$$

$$\left[\frac{d}{dw} F(w) = \frac{d}{dw}\left(\frac{1}{2\pi i} \oint_\Gamma \frac{F(z)}{z-w}\, dz\right) = \frac{1}{2\pi i} \oint_\Gamma \left(\frac{d}{dw} \frac{1}{z-w}\right) F(z)\, dz = \right.$$

$$\left. = \frac{1}{2\pi i} \oint_\Gamma \frac{F(z)}{(z-w)^2}\, dz \right].$$

Thus:

$$\ell\left(\frac{x(z_0+h)-x(z_0)}{h}\right)-\frac{d}{dz}\left(\ell(x(z))\right)\Big|_{z=z_0}=$$

$$=\frac{1}{2\pi i}\oint_\Gamma\left[\frac{1}{\cancel{h}}\cdot\frac{\cancel{h}}{(z-(z_0+h))(z-z_0)}-\frac{1}{(z-z_0)^2}\right]\ell(x(z))\,dz.=$$

$$=\frac{1}{2\pi i}\oint_\Gamma\frac{h}{(z-z_0)^2(z-(z_0+h))}\,\ell(x(z))\,dz.$$

$$\left|\ell\left(\frac{x(z_0+h)-x(z_0)}{h}\right)-\frac{d}{dz}\ell(x(z))\Big|_{z=z_0}\right|\le$$

$$\le\frac{1}{2\pi}\underbrace{\max_{z\in\Gamma}|\ell(x(z))|}\cdot|h|\int_\Gamma\frac{dz}{|z-z_0|^2\cdot|z-(z_0+h)|}$$

① $\forall \ell\in\mathcal{U}'$, $\{\ell(x(z)),z\in\Gamma\}$ is bounded.

Think of $\{x(z),z\in\Gamma\}$ as a family of operators acting on $\mathcal{U}'$.

By uniform boundedness theorem:

$\{\|x(z)\|,z\in\Gamma\}$ is bounded.

$\longrightarrow \sup_{z\in\Gamma}\|x(z)\|=:M<\infty.$

Fix $h_1, h_2 \in \mathbb{C}$, $|h_1| < \frac{r}{2}$, $|h_2| < \frac{r}{2}$

② Take. $l \in \mathcal{U}'$, $\|l\| = 1$ s.t.

$$\left\| \frac{x(z_0+h_1)-x(z_0)}{h_1} - \frac{x(z_0+h_2)-x(z_0)}{h_2} \right\| =$$

$$= l\left(\frac{x(z_0+h_1)-x(z_0)}{h_1} - \frac{x(z_0+h_2)-x(z_0)}{h_2} \right). \leq$$

$$\leq \left| l\left(\frac{x(z_0+h_1)-x(z_0)}{h_1} \right) - \frac{d}{dz} l(x(z)) \Big|_{z=z_0} \right| + \left| l\left(\frac{x(z_0+h_2)-x(z_0)}{h_2} \right) - \frac{d}{dz} l(x(z)) \Big|_{z=z_0} \right|$$

$$\leq \frac{1}{2\pi} M \cdot \left[|h_1| \int_\Gamma \frac{dz}{\underbrace{|z-z_0|^2}_{r} \cdot \underbrace{|z-(z_0+h_1)|}_{\geq r-|h_1| \geq \frac{r}{2}}} + |h_2| \int_\Gamma \frac{dz}{\underbrace{|z-z_0|^2}_{r} \cdot \underbrace{|z-(z_0+h_2)|}_{\geq \frac{r}{2}}} \right] \leq$$

$$\leq \frac{1}{\cancel{2\pi}} M \cdot \left[|h_1| \cdot \frac{\cancel{2\pi} \cancel{r}}{r^2 \, \frac{\cancel{r}}{2}} + |h_2| \frac{\cancel{2\pi} r}{r^2 \cdot \frac{r}{2}} \right] = \frac{4M}{r^2} \left[|h_1| + |h_2| \right].$$

$\Rightarrow$ For any $h_n \to 0$, $\left(\dfrac{x(z_0+h_n) - x(z_0)}{h_n} \right)_n$ is Cauchy hence convergent.

$$\Rightarrow \exists! \lim_{h \to 0} \frac{1}{h} \left(x(z_0+h) - x(z_0) \right). \text{ in } \mathcal{U}.$$

$\Rightarrow x : D \to \mathcal{U}$ is strongly analytic

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Proof of Gelfand Formula

(1) claim: $\lim\limits_{n\to\infty} \|x^n\|^{1/n}$ exists. Assume $x \neq 0$.

Note: $\|x^{n+m}\| \le \|x^n\| \cdot \|x^m\|$

Let $\alpha_n := \log \|x^n\|$.

$\|x^n\| \le \|x\|^n \longrightarrow \dfrac{\alpha_n}{n} \le \log \|x\|. \Rightarrow \limsup\limits_{n\to\infty} \dfrac{\alpha_n}{n} \le \log\|x\|$

$\alpha_{n+m} \le \alpha_n + \alpha_m$

Fix m. Let $n = k\cdot m + r$, $r \in \{0, 1, 2, \dots m-1\}$, $k \ge 0$.

$$\alpha_n = \alpha_{k\cdot m + r} \le \alpha_{k\cdot m} + \alpha_r \le k\cdot\alpha_m + \alpha_r.$$

$$\frac{\alpha_n}{n} \le \frac{k}{k\cdot m + r}\alpha_m + \frac{\alpha_r}{n} \le \frac{\alpha_m}{m} + \frac{\sup\limits_{0\le r\le m-1} \alpha_r}{n} \quad (\text{when } \alpha_m > 0).$$

$$\le \frac{k}{k+1}\cdot\frac{\alpha_m}{m} + \dots \quad (\text{when } \alpha_m < 0).$$

$$\Rightarrow \limsup_{n\to\infty} \frac{\alpha_n}{n} \le \frac{\alpha_m}{m} \longrightarrow \underbrace{\limsup_{n\to\infty} \frac{\alpha_n}{n}} \le \underbrace{\liminf_{m\to\infty} \frac{\alpha_m}{m}}$$

$\Longrightarrow$ Must be equal.

$\Rightarrow \lim\limits_{n\to\infty} \dfrac{\alpha_n}{n}$ exists and, is either finite. $\longrightarrow \lim\limits_{n\to\infty} \|x^n\|^{1/n} > 0$

or $-\infty$. $\longrightarrow \lim\limits_{n\to\infty} \|x^n\|^{1/n} = 0$.

—.

(2). Claim: $r(x) = \lim_{n\to\infty} \|x^n\|^{1/n}$.

Let. $\lambda = \lim_{n\to\infty} \|x^n\|^{1/n} \geq 0$.

(i) Want to show $\lambda \geq r(x)$: If $z \in \mathbb{C}$, $|z| > \lambda$

Then. want to show $z \in \rho(x)$. $\Leftrightarrow$ $z\cdot 1 - x$ invertible.

$$\underbrace{z\cdot 1 - x = z\cdot\left(1 - \frac{1}{z}x\right)}_{\text{invertible iff}} \quad 1 - \frac{1}{z}x \ \underline{\text{invertible}}$$

$\Leftarrow$ $\left[\sum_{n\geq 0} \frac{1}{z^n} x^n \text{ converges in } \mathcal{U}, \right]$. $\longleftrightarrow$ $\sum_{n\geq 0} a_n w^n$, $w = \frac{1}{z}$, $a_n = x^n$

$\Updownarrow$

$\frac{1}{|z|} <$ convergence radius of this series:

$$\frac{1}{\lim_{n\to\infty} \|a_n\|^{1/n}} = \frac{1}{\lim_{n\to\infty} \|x^n\|^{1/n}} = \frac{1}{\lambda}$$

$\Updownarrow$

$|z| > \lambda$.

<u>Alternate (shorter) proof:</u> <u>Claim:</u> If $z \in \sigma(x) \Rightarrow z^n \in \sigma(x^n)$, $\forall n \geq 1$ integer.

<u>Why:</u> $x^n - z^n \cdot 1 = (x - z\cdot 1)(x^{n-1} + z\cdot x^{n-2} + \dots + z^{n-1}\cdot 1)$

If $z\in\sigma(x)$:

<u>Claim</u> $\Rightarrow$ $|z^n| \leq \|x^n\| \Rightarrow |z| \leq \|x^n\|^{1/n} \Rightarrow |z| \leq \lim \|x^n\|^{1/n} = \lambda$.

II. $|z| > \lim_{n\to\infty} \|x^n\|^{1/n}$

$$\Rightarrow \exists N : \rho = \frac{\|x^N\|}{|z^N|} < 1, \quad \forall n \geq N : \underline{\frac{\|x^n\|}{|z|^n} \leq \rho^{n-N} \cdot C_{N,x}}.$$

$n = k.N + r.$

$$\longrightarrow \quad \sum_{n\geq 0} \frac{1}{z^n} x^n \text{ is convergent.}$$

ii).

$$R_x(z) = \frac{1}{z}\left(1 - \frac{1}{z}x\right)^{-1} \dashrightarrow \text{analytic around } \infty.$$

$$w = \frac{1}{z} \qquad (1 - wx)^{-1} \dashrightarrow \text{analytic at } 0.$$

$\downarrow$