Introduction: Metric Spaces, Normed Vector Spaces. (1)

Let X be a set.

Definition A function $d : X \times X \to \mathbb{R}$ is called **distance** on X. if it satisfies: $\forall x, y, z \in X$:

1) (Positivity): i) $d(x,y) \geq 0$

ii) $d(x,y) = 0$ iff (if and only if) $x = y$.

2) (Symmetry) $d(x,y) = d(y,x)$

3) (Triangle Inequality) $d(x,y) \leq d(x,z) + d(z,y)$.

Definition. (X, d). is called a metric space.

Remark. If. $d : X \times X \to \mathbb{R}$ satisfies only (i), (2), (3), [$d(x,x) =$

then d is called a semi-distance.

How to change semi-distance d into a distance $\hat{d}$?

Consider the equivalence relation: $x, y \in X$

$$x \sim y \quad \text{iff} \quad d(x,y) = 0.$$

$$\longrightarrow \quad \hat{X} = X/\sim \;.\; \hat{x} \in \hat{X} : \quad \hat{x} = \{ y \in X : d(x,y) = 0 \}.$$

Define: $\hat{d} : \hat{X} \times \hat{X} \to \mathbb{R}, \quad \hat{d}(\hat{x}, \hat{y}) = \inf\limits_{\substack{x \in \hat{x} \\ y \in \hat{y}}} d(x,y).$

Show: $\hat{d}(\hat{x}, \hat{y}) = d(x,y)$ & $(\hat{X}, \hat{d})$ is a metric space

Examples.

$$\underline{X} = C[0,1] = \{ f : [0,1] \to \mathbb{C} \,,\ f \text{ continuous} \}.$$

For, $0 < p \le \infty$:

$$d_p(f,g) = \begin{cases} \int_0^1 |f(x) - g(x)|^p \, dx \,, & 0 < p < 1. \\ \left(\int_0^1 |f(x) - g(x)|^p \, dx \right)^{1/p}, & 1 \le p < \infty. \\ \max_{x \in [0,1]} |f(x) - g(x)| \,, & p = \infty. \end{cases}$$

Exercise : Show $(\underline{X}, d_p)$ is a metric space.

$$L^p[0,1] = \left\{ f : [0,1] \to \mathbb{C} \,,\ f \text{ measurable} : \begin{array}{l} \int_0^1 |f(x)|^p \, dx < \infty, \text{ if } p < \infty \\ \operatorname{ess\,sup}_{x \in [0,1]} |f(x)| < \infty \,, \text{ if } p = \infty \end{array} \right\} / \text{a.e.}$$

$(L^p[0,1], d_p)$ is a metric space.

Note : $C[0,1] \subset L^p[0,1]$, for every $0 < p \le \infty$.

For $0 < p < \infty$, $C[0,1]$ is dense in $L^p[0,1]$.

Convergence in Metric Spaces.

Def A sequence $(x_n)_{n\geq 1}$ of (X, d) metric space is said convergent to $z \in X$ if:

$\forall \varepsilon > 0 \ \exists N = N(\varepsilon) \geq 1 \ \forall n \geq N, \quad d(x_n, z) < \varepsilon$.

Def. A sequence $(x_n)_{n\geq 1}$ in (X, d) is said convergent (in X) if there exists $z \in X$ s.t. $(x_n)_{n\geq 1}$ is convergent to z.

Def. A sequence $(x_n)_{n\geq 1}$ in (X, d) is said Cauchy if:

$\forall \varepsilon > 0 \ \exists N = N(\varepsilon) \ \forall n, m \geq N, \quad d(x_n, x_m) < \varepsilon$.

Remark.

If $(x_n)_{n\geq 1}$ is convergent then $(x_n)_{n\geq 1}$ is Cauchy.

Definition. A metric space (X, d) is said complete. if every Cauchy sequence is convergent in X.

Examples: For every $0 < p \leq \infty$,

$(L^p[0,1], d_p)$ is a complete metric space.

Let $(\underline{X}, d)$ be a metric space. Denote $B_r(x) = \{y \in \underline{X} : d(x,y) < r\}$

Let $A \subset \underline{X}$ be a subset of $\underline{X}$. ball of radius r centered at x

<u>Definition</u> The set A is called:

1) <u>open</u>. if: $\forall x \in A \ \exists r > 0$ s.t. $B_r(x) \subset A$

2) <u>closed</u> if: $\underline{X} \setminus A$ is open.

3) <u>dense(in $\underline{X}$)</u> if $\forall \varepsilon > 0 \ \forall x \in \underline{X} \ \exists y \in A$ s.t. $d(x,y) < \varepsilon$

4) <u>compact</u>: If. $(U_\alpha)_{\alpha \in \mathcal{A}}$ is a family of open sets of $\underline{X}$ such that $A \subset \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ then there exists a finit subset $\{\alpha_1, \alpha_2, \dots \alpha_N\} \in \mathcal{A}$ such that $A \subset \bigcup_{k=1}^{N} U_{\alpha_k}$

5) <u>a neighborhood of x</u> if there exists an open set $B \subset \underline{X}$ such that: $x \in B \subset A$

<u>Definitions</u>.

1) A point $x \in \underline{X}$ is called a <u>limit point</u> of a set $A \subset \underline{X}$ if:

$$\forall r > 0, \quad B_r(x) \cap (A \setminus \{x\}) \neq \phi.$$

2) A point $x \in \underline{X}$ is called an <u>interior point</u> of $A \subset \underline{X}$ if A is a neighborhood of x $\iff \exists r > 0$ s.t. $B_r(x) \subset A$.

Definitions:

Let $A \subset X$, (X, d) metric space.

1). The set $\mathring{A}$ (or A^{int}) is called the interior of A if $\mathring{A}$ is the collection of all interior points.

2) The set $\bar{A}$ (or $cl(A)$) formed by A union with the set of limit points is called the closure of A.

Remark:

$$\mathring{A} = \bigcup_{\substack{\theta \subset A \\ \theta \text{ open}}} \theta \quad , \quad \bar{A} = \bigcap_{\substack{A \subset \mathcal{C} \\ \mathcal{C} \text{ closed}}} \mathcal{C}.$$

$(L^p[0,1], d_p)$ is a complete metric space.

$C[0,1]$ is dense in $L^p[0,1]$, $0 < p < \infty$.

$(C[0,1], d_\infty)$ is a complete metric space.