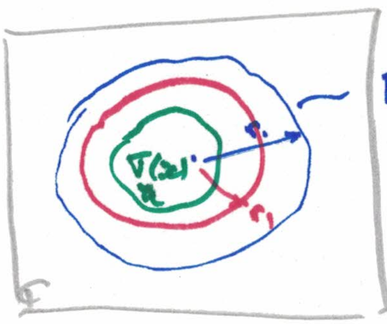
Holomorphic Calculus (3)

(Proof of previous Theorem)

(e) If $f(z) = \sum_{n \geq 0} a_n z^n$, $a_n \in \mathbb{C}$ has radius of convergence $r > r_{\mathcal{U}}(x)$

then $f(x) = \sum_{n\geq 0} a_n x^n$, $f(x) \in \mathcal{U}$.

$\underline{\underline{\text{Pf}}}$: follows from (d):



$B_r(0)$.

$\left.\begin{array}{l} f\big|_{B_r(0)} : B_r(0) \to \mathbb{C} \text{ is holomorphic.} \\ \sigma(x) \subset B_r(0) \end{array}\right| \Rightarrow f \in \mathcal{F}(x)$

Take $r(x) < r_1 < r$ $\Rightarrow$ $\left(\sum_{k=0}^{n} a_k z^k\right)_n \longrightarrow f$, uniformly on $U_{r_1}(0) = \overline{B_{r_1}(0)}$.

By part (d) $\Rightarrow$ $\left(\sum_{k=0}^{n} a_k x^k\right)_n \longrightarrow f(x)$, in $\mathcal{U}$-norm.

(f). If $\lambda \in \rho(x)$ and $f(z) = \dfrac{1}{z-\lambda}$ then $f(x) = (x - \lambda \cdot 1)^{-1}$

$\tilde{f}(z) = \dfrac{1}{\lambda - z}$.

$\tilde{f}(x) = (\lambda \cdot 1 - x)^{-1} = R_x(\lambda)$.

$\underline{\underline{\text{Proof}}}$: Define $g(z) = z - \lambda$.. $f(z)\cdot g(z) = 1$, $\forall z \in \mathbb{C}$

$\Rightarrow$ $f(x) \cdot g(x) = \mathbb{1} = g(x)\cdot f(x)$ $\Rightarrow$ $f(x) = (g(x))^{-1} = (x - \lambda\cdot 1)^{-1}$

$g(x) = x - \lambda \cdot 1$

(g) $\to$ look for details: (S

Consequences of previous Theorem:

Theorem [Spectral Mapping Theorem, version 2].

Let $\mathcal{U}$ be a Banach algebra with identity and $x \in \mathcal{U}$.

Let $f : N_f \to \mathbb{C}$ be a holomorphic function on open set $N_f \subset \mathbb{C}$,

s.t. $\sigma_{\mathcal{U}}(x) \subset N_f$.

Then: $\sigma(f(x)) = f[\sigma(x)]$.

(where $f[S] = \{ f(\lambda), \lambda \in S \}$, $S \subset \mathbb{C}$).

Proof.

(a). Suppose. $\lambda \notin f[\sigma(x)] \iff \lambda \in \mathbb{C} \setminus f[\sigma(x)]$.

We show that $\lambda \notin \sigma(f(x)) \iff f(x) - \lambda \cdot 1$ is invertible in $\mathcal{U}$.

$\sigma(x)$ compact $\to$ $f[\sigma(x)]$ compact non-empty.
$\neq \phi$

Let $V \subset \mathbb{C}$ open set s.t. $f[\sigma(x)] \subset V$, $\lambda \notin V$.

Let $U \subset \mathbb{C}$, $U = f^{-1}(V)$, $\sigma(x) \subset U$.

Define $g : z \longmapsto \dfrac{1}{f(z) - \lambda}$. $g : U \to \mathbb{C}$ is holomorphic.

$x \overset{g}{\longmapsto} g(x) \in \mathcal{U}.$: $(f(z)-\lambda)\cdot g(z) = 1, \ \forall z \in U.$

And : $(f(x)-\lambda\cdot 1)\cdot g(x) = g(x)\cdot(f(x)-\lambda\cdot 1) = \mathbb{1}$

$\Rightarrow$ $f(x)-\lambda\cdot 1$ invertible $\Rightarrow \lambda \notin \sigma(f(x))$.

(b). Let $\lambda \in f[\sigma(x)]$.

<u>Want</u>: $\lambda \in \sigma(f(x))$.

$\lambda \in f[\sigma(x)] \longrightarrow \lambda = f(\mu)$, for some $\mu \in \sigma(x)$.

Define $g : N_f \longrightarrow \mathbb{C}$, $g(z) = \dfrac{f(z)-\lambda}{z-\mu} = \dfrac{f(z)-f(\mu)}{z-\mu}$.

g is holomorphic on N_f : μ is a removable singularity.

$$g(z)\cdot(z-\mu) = (z-\mu)\cdot g(z) = f(z)-\lambda, \ \forall z \in N_f$$

$$\Downarrow$$

$$g(x)\cdot(x-\mu\cdot 1) = (x-\mu\cdot 1)\, g(x) = f(x)-\lambda\cdot 1.$$

If. $\lambda \notin \sigma(f(x)) \Rightarrow$ $f(x)-\lambda\cdot 1$ is ~~not~~ invertible. $\Rightarrow$

$\Rightarrow$ $x-\mu\cdot 1$ is invertible. $\Rightarrow \mu \notin \sigma(x)$. Contradiction.

$\blacksquare$

<u>Theorem</u> Let $\mathcal{U}$ be a Banach algebra with identity and $x \in \mathcal{U}$.

Let $\sigma(x) = \sigma_1 \cup \sigma_2$, where $\sigma_1 \cap \sigma_2 = \phi$, $\sigma_1 \neq \phi$, $\sigma_2 \neq \phi$.

Let U_1, U_2 be two open sets s.t.: $U_1 \cap U_2 = \phi$, $\sigma_1 \subset U_1$, $\sigma_2 \subset U_2$

Let $f : U \to \mathbb{C}$, $U = U_1 \cup U_2$, $f(z) = \begin{cases} 1, & z \in U_1 \\ 0, & z \in U_2 . \end{cases}$

Then: (1) $f \in \mathcal{F}(x)$, i.e. holomorphic on open set $U \supset \sigma(x)$.

(2) Let $p = f(x)$. Then p is a <u>projection</u>, i.e. <u>idempotent</u>

i.e., $p^2 = p$. And $p \neq 0$, $p \neq 1$.

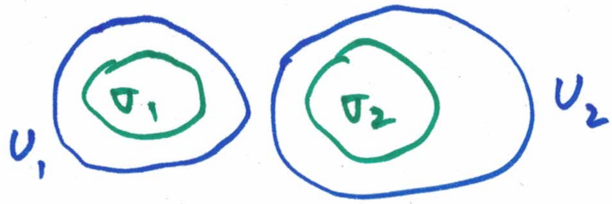
(3) For any $\alpha, \beta \in \mathbb{C}$ we have:

$$\sigma(x \cdot p + \alpha \cdot (1-p)) = \sigma_1 \cup \{\alpha\}.$$

$$\sigma(x \cdot (1-p) + \beta \cdot p) = \sigma_2 \cup \{\beta\}.$$

<u>Proof</u>.

(1).



(2) Let $p = f(x)$, Note: $f(z) \cdot f(z) = f(z)$, $\forall z \in U$

$$\Rightarrow p^2 = p. \quad : \text{idempotent}.$$

But $f \neq 1$ constant on $\sigma(x)$ $\Rightarrow$ $p \neq 1$

$\neq 0$ constant on $\sigma(x)$ $\qquad p \neq 0$.

(7)

$\sigma(x\cdot p+\alpha\cdot(1-p))=\,?$

let $g: U \to \mathbb{C}$, $g(z)=z\cdot f(z)+\alpha\cdot(1-f(z))$

g holomorphic.

Note: $g(x)=x\cdot f(x)+\alpha\cdot(1-f(x))=x\cdot p+\alpha\cdot(1-p)$

$\Rightarrow\ \sigma(g(x))=g\{\sigma(x)\}=g\{\sigma_1\cup\sigma_2\}=\sigma_1\cup\{\alpha\}.$

Similarly: $h: U\to\mathbb{C}$, $h(z)=z\cdot(1-f(z))+\beta\cdot f(z)$.

$\Rightarrow\ h(x)=x\cdot(1-p)+\beta\cdot p$

$\sigma(x\cdot(1-p)+\beta\cdot p)=h[\sigma(x)]=\{\beta\}\cup\sigma_2$ —.

Corollary [Spectral localization Theorem].

Let $A\in B(\mathfrak{X})$, where $\mathfrak{X}$ is a Banach space. Suppose $\sigma(A)=\sigma_1\cup\sigma_2$, σ_1,σ_2 closed sets. where $\sigma_1\cap\sigma_2=\phi$, $\sigma_1\neq\phi$, $\sigma_2\neq\phi$. Let U_1, U_2 be two open sets in $\mathbb{C}$ s.t. $\sigma_1\subset U_1$, $\sigma_2\subset U_2$, $U_1\cap U_2=\phi$. (disjoint open sets). Let $U=U_1\cup U_2$ and $f: U\to\mathbb{C}$, $f(z)=\begin{cases}1, & z\in U_1\\ 0, & z\in U_2\end{cases}$. Let $P=f(A)$.

Then:

1) $A\cdot P|_{Ran(P)}: Ran(P)\longrightarrow Ran(P)$

$A\cdot(1-P)|_{Ran(I-P)}: Ran(1-P)\longrightarrow Ran(1-P)$

and,

2) $\sigma(A\cdot P|_{Ran(P)})=\sigma_1$

$\sigma(A\cdot(1-P)|_{Ran(1-P)})=\sigma_2$ —.

Proof

let $E = \mathrm{Ran}(P)$, $F = \mathrm{Ran}(I-P)$

Claim: E, F closed spaces in X, $E \cap F = \{0\}$, $X = E \oplus F$.

$P^2 = P \longrightarrow$ If. $(x_n)_{n \geq 1}$, $x_n \in E$. $x_n \to y$, in X.

Then: $x_n = P(x_n) \xrightarrow{n\to\infty} y = P(y) \Rightarrow \underline{y \in E}$.

$(1-P)^2 = 1-P \longrightarrow$ same for F.

If $x \in E \cap F \longrightarrow x = Px = (1-P)x \Rightarrow \underline{x = 0}$.

(1). $A \cdot P = g(A)$. , where $g : U \to \mathbb{C}$.

$$g(z) = \begin{cases} z, & z \in U_1 \\ 0, & z \in U_2 \end{cases} = z \cdot f(z)$$

$\Downarrow$

$g(A) = A \cdot f(A) = A \cdot P \Rightarrow A \cdot P = P \cdot A \not\Rightarrow$

and. $A \cdot (1-P) = (1-P) \cdot A$.

$\Rightarrow A \cdot P\big|_{\mathrm{Ran}(P)} : \mathrm{Ran}(P) \to \mathrm{Ran}(P)$

$A \cdot (1-P)\big|_{\mathrm{Ran}(I-P)} : \mathrm{Ran}(1-P) \to \mathrm{Ran}(1-P)$.

(2), Fact: $B_1 : \mathrm{Ran}(P) \to \mathrm{Ran}(P)$

$B_2 : \mathrm{Ran}(1-P) \to \mathrm{Ran}(1-P)$.

let $B = B_1 \oplus B_2$, i.e. $B(x) = B_1 x_1 + B_2 x_2$

where $x = x_1 + x_2$, $x_1 \in \mathrm{Ran}(P)$

$x_2 \in \mathrm{Ran}(1-P)$.

Then $\sigma(B) = \sigma(B_1) \cup \sigma(B_2)$

where $B_1 \in B(\mathrm{Ran}(P))$, $B_2 \in B(\mathrm{Ran}(1-P))$

Reason:

If. $C = C_1 \oplus C_2$, $C_1 \in B(\mathrm{Ran}(P))$

$C_2 \in B(\mathrm{Ran}(1-P))$.

$$B \cdot C = (B_1 C_1) \oplus (B_2 C_2).$$

$\Leftrightarrow$ $B - \lambda \cdot I$ invertible $\Leftrightarrow$ $B_1 - \lambda \cdot I|_{\mathrm{Ran}\,P}$ invertible.

and. $B_2 - \lambda \cdot I|_{\mathrm{Ran}(P)}$ invertible.

$\Rightarrow$ $\rho(B) = \rho(B_1) \cap \rho(B_2)$

$\Rightarrow$ $\sigma(B) = \sigma(B_1) \cup \sigma(B_2)$.

Pick $\alpha \notin \sigma$. Construct $h : U \to \mathbb{C}$, $h(z) = \begin{cases} z, & z \in U_1 \\ \alpha, & z \in U_2 \end{cases}$

$\rightarrow$ $\sigma(h(A)) = \sigma_1 \cup \{\alpha\}$. $h(A) = A \cdot P \oplus \alpha \cdot (1-P)$

$\Rightarrow$ $\sigma(A \cdot P|_{\mathrm{Ran}\,P}) = \sigma_1$

similarly, $\sigma(A \cdot (1-P)|_{\mathrm{Ran}(1-P)}) = \sigma_2$.