L24

Algebras of Operators on Hilbert Spaces.

- Von Neumann Algebra (W^*-algebra).
- Polar Decomposition.
- Spectral Calculus.

Today: General Results.

Let $(H, \langle \cdot, \cdot \rangle)$ denote a Hilbert space.

$$B(H) = \{ T: H \to H,\ T \text{ linear and bounded}: \|T\| = \sup_{\substack{\|x\|=1 \\ x \in H}} \|Tx\| < \infty \}$$

Definition Given $T \in B(H)$, its adjoint denoted T^* is the bounded operator, $T^* \in B(H)$ such that,

$$\forall x, y \in H, \quad \langle Tx, y \rangle = \langle x, T^* y \rangle.$$

Properties. Assume $A_1, A_2, A \in B(H)$, $a_1, a_2 \in \mathbb{C}$. Then:

(i) $(a_1 A_1 + a_2 A_2)^* = \bar{a}_1 A_1^* + \bar{a}_2 A_2^*$

(ii). $\|A^*\| = \|A\|$.

(iii) $(A^*)^* = A$.

(iv) $(A_1 \cdot A_2)^* = A_2^* \cdot A_1^*$ ← NOTE ALSO.

Definitions. Let $A : H \to H$ be a bounded operator.

① A is called self-adjoint : $A^* = A$

② A is called unitary : $A^* A = A \cdot A^* = I_H$ (identity operator) $(A^{-1} = A^*)$.

③ A is called normal : $A^* A = A A^*$

④ A is called projection (idempotent) : $A^2 = A$

⑤ A is called orthogonal projection : $A^2 = A$ and $A^* = A$.

Remarks:

If $T : H_1 \to H_2$ is a bounded operator between two Hilbert spaces.

Then:

1) $T^* : H_2 \to H_1$, is the adjoint operator:

$$\forall x \in H_1,\ \forall y \in H_2,\quad \langle Tx, y \rangle_{H_2} = \langle x, T^* y \rangle_{H_1}$$

2) T is called unitary if:

$$T^{-1} = T^* \iff T T^* = 1_{H_2},\quad T^* T = 1_{H_1}$$

(identity operator).

3). T is called an isometry if: $\forall x \in H_1,\ \|Tx\|_{H_2} = \|x\|_{H_1}$

(T is unitary) iff. (T isometry and T surjective.)

4) T is called a coisometry if T^* is an isometry.

<u>Proposition</u> ① Assume $P \in B(H)$, $P^2 = P$ (is a projection).

(1) $Q = 1 - P$ is a projection.

(2) $\mathrm{Ran}(P)$, $\mathrm{Ran}(Q)$ are closed subspaces of H.

(3) $\mathrm{Ran}(P) \cap \mathrm{Ran}(Q) = \{0\}$.

(4) $H = \mathrm{Ran}(P) \oplus \mathrm{Ran}(Q)$

(5) $|||x||| = \|Px\| + \|Qx\|$: $|||\cdot|||$ is an equivalent norm to $\|\cdot\|$.

$\Rightarrow \exists A, B > 0$ s.t. $A \cdot \|x\| \le \|Px\| + \|Qx\| \le B \cdot \|x\|$.

(6) If $E, F \subset H$ are two closed subspaces of H s.t. $E \cap F = \{0\}$, $E \oplus F = H$ then there exists a unique projection $P : H \to H$ such that $\mathrm{Ran}(P) = E$, $\mathrm{Ran}(1-P) = F$.

<u>Remark</u>:

(5):

i bounded $\Rightarrow \|Px\| + \|Qx\| \le (\|P\| + \|Q\|) \cdot \|x\|$.

$$H \xrightarrow{i} \mathrm{Ran}(P) \oplus \mathrm{Ran}(Q) := \{u \oplus v : u \in \mathrm{Ran}\,P,\ v \in \mathrm{Ran}\,Q\}.$$

$$x \longmapsto (Px, Qx).$$

$\|u \oplus v\| = \|u\| + \|v\|$!

By inverse mapping theorem: i^{-1} is also bounded.

$$\|x\| \le \underline{C} \cdot (\|Px\| + \|Qx\|),\ \forall x.$$

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(6). ...

<u>Proposition ②</u>. Assume $P, Q \in B(\mathcal{H})$ are idempotents that commute: $PQ = QP$, $P^2 = P$, $Q^2 = Q$.

(a) $R = P \cdot Q$ is also a projection.

(b) $\mathrm{Ran}(R) = \mathrm{Ran}(P) \cap \mathrm{Ran}(Q)$.

(c) $\ker(R) = \ker(P) + \ker(Q)$

(d) $P + Q - PQ$ is the projection onto $\mathrm{Ran}(P) + \mathrm{Ran}(Q)$.

In particular $PQ = 0 \iff \mathrm{Ran}(P) \cap \mathrm{Ran}(Q) = \{0\}$.

<u>Pf./Sketch</u>.

(a) $R^2 = PQPQ = P^2Q^2 = PQ = R$.

(b). Let $y \in \mathrm{Ran}(P) \cap \mathrm{Ran}(Q)$: $y = Px_1$, $y = Qx_2$, for some x_1, x_2.

$$\left.\begin{aligned} Rx_2 &= PQx_2 = Py = y. \\ Rx_1 &= QPx_1 = Qy = y. \end{aligned}\right\} \Rightarrow y \in \mathrm{Ran}(R).$$

$$\left.\begin{aligned} \mathrm{Ran}(R) &= \mathrm{Ran}(PQ) \subset \mathrm{Ran}(P) \\ &= \mathrm{Ran}(QP) \subset \mathrm{Ran}(Q) \end{aligned}\right\} \Rightarrow \mathrm{Ran}(R) \subset \mathrm{Ran}(P) \cap \mathrm{Ran}(Q)$$

(c). $x \in \ker(R)$: $Rx = 0$: $PQx = QPx = 0$.

$$x = Px + (1-P)x = P(Q + (1-Q))x + (1-P)(Q + (1-Q))x =$$

$$= \underbrace{PQx}_{=0.} + P(1-Q)x + (1-P)Qx + (1-P)(1-Q)x.$$

$\Rightarrow x \in \mathrm{Ran}(1-P) + \mathrm{Ran}(1-Q) = \ker(Q) + \ker(P)$

We obtained: $\ker(R) \subset \ker(P) + \ker(Q)$

Conversely: if $x = x_1 + x_2$, with $Px_1 = 0$, $Qx_2 = 0$.

then: $Rx = PQx_1 + PQx_2 = Q\underbrace{Px_1}_{=0} + P\underbrace{Qx_2}_{=0} = 0$.

$\Rightarrow x \in \ker(R)$.

(d) ...

Construction. If $A \in B(H)$, then.

$$B = \frac{1}{2}(A + A^*), \quad C = \frac{1}{2i}(A - A^*)$$

are self-adjoint operators and. $A = B + i \cdot C$.

Theorem 1 let $A \in B(H)$. Then:

(1) $\|A^* A\| = \|A\|^2$

(2) If A is normal then $\underset{B(H)}{r(A)} = \|A\|$ and. $\|A^n\| = \|A\|^n$

Theorem 2. Assume $P \in B(H)$ is an idempotent, i.e. $P^2 = P$, $P \neq 0$

Then the following are equivalent:

(i) P is an orthogonal projection, i.e. $P^* = P$

(ii) $\|P\| = 1$

(iii) $\operatorname{Ran}(P) \perp \operatorname{Ran}(1-P)$.

Proof of Theorem 1

(1) $\|A^*A\| = \|A\|^2$.

$$\|A^*A\| \le \|A^*\| \cdot \|A\| = \|A\| \cdot \|A\| = \|A\|^2.$$

Cauchy-S. ineq

$$\|A^*\|^2 = \sup_{\|x\|=1} \|Ax\|^2 = \sup_{\|x\|=1} \langle Ax, Ax\rangle = \sup_{\|x\|=1} |\langle A^*Ax, x\rangle| \le$$

$$\le \sup_{\|x\|=1} \left[\|A^*Ax\| \cdot \|x\|\right] = \sup_{\|x\|=1} \|A^*Ax\| = \|A^*A\|.$$

$$\Rightarrow \quad \|A^*A\| = \|A\|^2.$$

(2) Assume A is normal: $A^*A = AA^*$

By Gelfand's Formula:

$$\underset{B(H)}{r}(A) = \lim_{n\to\infty} \|A^n\|^{1/n} = \lim_{n\to\infty} \|A^{2^n}\|^{\frac{1}{2^n}}$$

Note: let $B = A^*A$:

1) $$\|A^{2^n}\|^2 \underset{\text{by part(1)}}{=} \|(A^{2^n})^* \cdot (A^{2^n})\| = \|\underbrace{A^*A^*\cdots A^*}_{2^n \text{ times}} \underbrace{AA\cdots A}_{2^n \text{ times}}\| =$$

$$= \|(A^*A)^{2^n}\| = \|B^{2^n}\|.$$

2). $$\|B^2\| = \|B^* \cdot B\| \overset{\text{by part(1).}}{=} \|B\|^2 \; *$$

$$\|B^4\| = \cdots = \|B\|^4 \xrightarrow{\text{by induction}} \|B^{2^n}\| = \|B\|^{2^n}$$

3) $\|B\| = \|A^*A\| = \|A\|^2$

$$\Rightarrow \quad \|A^{2^n}\|^2 = \|B^{2^n}\| = \|B\|^{2^n} = \|A\|^{2^{n+1}}$$

$$\Rightarrow \quad \|A^{2^n}\| = \|A\|^{2^n}$$

$$\Rightarrow \quad r_{B(H)}(A) = \lim_{n\to\infty} \|A^{2^n}\|^{\frac{1}{2^n}} = \underline{\|A\|}.$$

On the other hand: $\|A^{2^n}\| = \|A\|^{2^n}$ $\not\Rightarrow$ $\|A^n\| = \|A\|^n$.

Since $\|A^{\alpha+\beta}\| \le \|A^\alpha\| \cdot \|A^\beta\|$ ...

Conclusion: $\boxed{\max_{\lambda\in\sigma(A)} |\lambda| = \|A\|.}$

Proof of Theorem 2: $P^2 = P$.

(i) $\Rightarrow$ (ii). : Assume $P^* = P$.

$$\|P\|^2 = \|P^*P\| = \|P^2\| = \|P\| \overset{P\neq 0}{\Rightarrow} \underline{\|P\| = 1}.$$

↑ by previous Theorem

(ii) $\Rightarrow$ (iii). Assume $\|P\| = 1$.

Let $u \in Ran(P)$, $v \in Ran(1-P)$.

(8).

For any $a \in \mathbb{C}$:

$$P(u+a\cdot v) = P(u) + a\cdot P(v) = u + a\cdot \underbrace{P(1-P)}_{=0} v = u.$$

$$\|u\| = \|P(u+a\cdot v)\| \leq \underbrace{\|P\|}_{1} \cdot \|u+a\cdot v\| = \|u+a\cdot v\|.$$

$$\|u\|^2 \leq \|u+a\cdot v\|^2 = \|u\|^2 + 2\,\mathrm{Re}\left(\bar{a}\cdot\langle u,v\rangle\right) + \|v\|^2\cdot|a|^2$$

If $\langle u,v\rangle \neq 0$:

Take $a = t\cdot \dfrac{\langle u,v\rangle}{|\langle u,v\rangle|}$, $t\in\mathbb{R}$.

$$\cancel{\|u\|^2} \leq \cancel{\|u\|^2} + 2t\cdot|\langle u,v\rangle| + \|v\|^2\cdot t^2$$

$$0 \leq \underbrace{2t\cdot|\langle u,v\rangle| + t^2\cdot\|v\|^2}_{\text{quadratic in } t,}$$

$\Rightarrow$ Real roots coincide $\Rightarrow$ $t_1 = t_2 = 0$

$\Rightarrow \underline{\underline{\langle u,v\rangle = 0}}$

$\Rightarrow$. $\mathrm{Ran}(P) \perp \mathrm{Ran}(1-P)$.

iii) $\Rightarrow$ (i) : Assume: $\langle Px, (1-P)y\rangle = 0$, $\forall x,y\in H$.

$$\langle Px,y\rangle = \langle Px, Py + (1-P)y\rangle = \langle Px,Py\rangle + \underbrace{\langle Px,(1-P)y\rangle}_{=0}$$

$$= \langle Px,Py\rangle$$

$$\langle x,Py\rangle = \langle Px + (1-P)x, Py\rangle = \langle Px,Py\rangle + \underbrace{\langle (1-P)x,Py\rangle}_{=0}$$

$$= \langle Px,Py\rangle.$$

$\Rightarrow \langle Px,y\rangle = \langle x,Py\rangle$, $\forall x,y\in H$ $\Rightarrow$ $P = P^*$. $\boxtimes$