L26

Polar Decomposition

Some Results about Self-adjoint operators.

Let H be a Hilbert space.

Proposition. If $A \in B(H)$, $A = A^*$ is a self-adjoint bounded operator then $\underbrace{\sigma(A)}_{} \subset \mathbb{R}$. Equivalently, $\mathbb{C} \setminus \mathbb{R} \subset \underbrace{\rho(A)}_{}$.

$\sigma_{B(H)}(A)$: spectrum of A $\qquad$ $\rho_{B(H)}(A)$: resolvent set of A.

Proof.

We show, if $\lambda, \mu \in \mathbb{R}$, $\mu \neq 0$ then $\lambda + i\mu \in \rho(A)$.

Equivalent: $\lambda + i\mu - A$ is invertible (in $B(H)$).

$\Updownarrow$

$\lambda + i\mu - A$ is injective. $\Leftrightarrow \ker(\lambda + i\mu - A) = \{0\}$

$\lambda + i\mu - A$ is surjective $\Leftrightarrow$ $\mathrm{Ran}(\lambda + i\mu - A) = H$.

Take $x \in H$,

$$\|(\lambda + i\mu - A)x\|^2 = \langle (\lambda + i\mu - A)x, (\lambda + i\mu - A)x \rangle =$$

$$= \langle i\mu x, i\mu x \rangle + \underbrace{\langle i\mu x, (\lambda - A)x \rangle + \langle (\lambda - A)x, i\mu x \rangle}_{0} + \langle (\lambda - A)x, (\lambda - A)x \rangle$$

$$= \mu^2 \cdot \|x\|^2 + \|(\lambda - A)x\|^2 \geq \mu^2 \cdot \|x\|^2$$

Hence: $\|(\lambda + i\mu - A)x\| \geq |\mu| \cdot \|x\|$, for any $\lambda, \mu \in \mathbb{R}$, $x \in H$.

Consequences: If $\mu \neq 0$.

1) $\ker(\lambda + i\mu - A) = \{0\}$, $\quad \ker(\lambda - i\mu - A) = \{0\}$.

2). $\mathrm{Ran}(\lambda + i\mu - A)$ is closed in H.

$\mathrm{Ran}(\lambda - i\mu - A)$ is closed in H.

(3) $z = \lambda + i\mu$,

$$\|(z-A)^{-1}\| \leq \frac{1}{|\mu|} = \frac{1}{|\mathrm{Im}(z)|}.$$

--- Once we establish that $z \in \rho(A)$.

(1) $\longrightarrow$ $\lambda + i\mu - A$, $\lambda - i\mu - A$ are injective.

Note: If $T \in B(H)$, $\overline{\mathrm{Ran}(T)} = \ker(T^*)^{\perp}$

(see Homework).

$$\overline{\mathrm{Ran}(\lambda + i\mu - A)} = \ker((\lambda + i\mu - A)^*)^{\perp} = \ker(\lambda - i\mu - A)^{\perp} =$$

$$= \{0\}^{\perp} = H.$$

$\downarrow$

(2) $\Rightarrow$ $\mathrm{Ran}(\lambda + i\mu - A) = H$.

$\boxtimes$

Since. $r(A) = \|A\|$, for $A = A^*$
spectral radius

$\Rightarrow$ $\sigma(A) \subset [-\|A\|, \|A\|]$.

<u>Proposition</u> (Corollary of the $\sqrt{\cdot}$ Lemma). Assume $A = A^*$, $A \in B(\mathcal{H})$.
The following are equivalent:

(1) $A \geq 0$

(2) $\sigma(A) \subset [0, \infty)$

<u>Sketch of Proof</u>:

$(1) \Rightarrow (2)$: $A \geq 0 \cdots \to \exists B = \sqrt{A} = B^*$

$A = B^2$

$\sigma(A) = \sigma(B^2) = \{ x^2, \; x \in \sigma(B) \subset \mathbb{R} \}. \subset [0, \infty).$

$(2) \Rightarrow (1)$: $\tilde{A} = \dfrac{A}{\|A\|}$: $\sigma(\tilde{A}) \subset [0, 1]$.

$\sigma(1 - \tilde{A}) \subset [0, 1]$.

$\|\tilde{A}\| = 1$, $\|1 - \tilde{A}\| \leq 1$

$\Rightarrow$ Use Lemma before theorem to construct

$$\mathcal{B} = \lim_{N \to \infty} p_N(\tilde{A}), \text{ in } B(\mathcal{H}), \quad B = B^*.$$

s.t. $\tilde{A} = B^2 \dashrightarrow \langle \tilde{A} x, x \rangle = \langle B^2 x, x \rangle =$

$= \|Bx\|^2 \geq 0 \Rightarrow \tilde{A} \geq 0$

$\Rightarrow A \geq 0.$

Partial Isometries.

Definition. An operator $U \in B(H)$ is called a partial isometry is:

(1) $U^* U$ is an orthogonal projection.

(2) $U U^*$ is an orthogonal projection.

Notations: $H_I = \text{Ran}(U^* U)$ is called the initial subspace of U

$H_F = \text{Ran}(U U^*)$ is called the final subspace of U.

(the "target subspace").

Proposition.

$\boxed{I}$ Assume $U \in B(H)$ is a partial isometry. Then:

(a) $H_I^{\perp} = \ker(U)$

(b) $H_F = \text{Ran}(U)$.

(c). $U\big|_{H_I} : H_I \longrightarrow H_F$ is a unitary map.

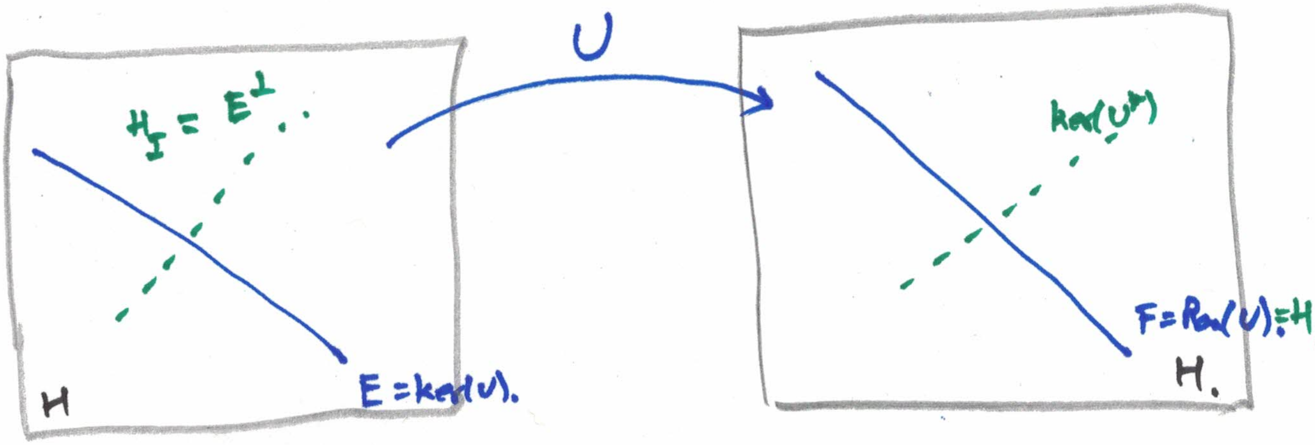
$\boxed{II}$ Assume. U satisfies: i) $\text{Ran}(U)$ is closed and

ii) $U\big|_{\ker(U)^{\perp}} : \ker(U)^{\perp} \longrightarrow \text{Ran}(U)$ is a unitary map.

Then U is partial isometry with initial space $H_I = \ker(U)^{\perp}$

final subspace $H_F = \text{Ran}(U)$.

Pictorially:



Sketch of Proof:

Assume. U is a partial isometry.

Then $U\big|_{\ker(U)^\perp} : \underset{\substack{\shortparallel \\ H_I}}{\ker(U)^\perp} \longrightarrow \mathrm{Ran}(U)$ is unitary:

$U^* : H \longrightarrow H, \quad \ker(U^*) = \mathrm{Ran}(U)^\perp$

$H/\ker(U^*). \cong (\ker(U^*))^\perp = ((\mathrm{Ran}(U))^\perp)^\perp = \mathrm{Ran}(U).$

$U^*\big|_{\ker(U^*)^\perp = \mathrm{Ran}(U)} : \mathrm{Ran}(U) \longrightarrow \mathrm{Ran}(U^*) = \ker(U)^\perp = H_I.$

U^*U is an orth. proj $: H \longrightarrow H_I$

$U^*U\big|_{H_I} : H_I \to H_I$ identically: $U^*U\big|_{H_I} = 1_{H_I}$

UU^* is an orth proj $: H \longrightarrow H_F$

$UU^*\big|_{H_F} = 1_{H_F}$

$\Longrightarrow U\big|_{H_I} : H_I \to H_F$ unitary.

——.

Theorem [The Polar Decomposition Theorem] Let $A \in B(H)$.

Then there exists a unique partial isometry U so that:

(a) $A = U \cdot |A|$

(b) $H_I(U) = \overline{Ran(|A|)} = \ker(A)^\perp$

(c) $H_F(U) = \overline{Ran(A)} = \ker(A^*)^\perp$

Intuition (1) $z \in \mathbb{C} \dashrightarrow z = e^{i\theta} \cdot \underbrace{|z|}_{\geq 0}$.

$[\, e^{i\theta} : \mathbb{C} \to \mathbb{C},\ (e^{i\theta})(w) = e^{i\theta} \cdot w$: unitary $\dashrightarrow$ partial isometry.

Remark, Assume, $A \in B(H)$, let $\mathcal{C}$ denote the C^*-algebra generated by $\{A, A^*\}$ (smallest C^*-algebra that includes A).

$$|A| = \sqrt{A^* A} \in \mathcal{C}$$

$$A \in \mathcal{C}$$

However: U may not be in $\mathcal{C}$. Instead U always belong to the W^*-algebra generated by $\mathcal{C}$.

Intuition (2): $A \in \mathbb{C}^{n \times n}$, $\dashrightarrow$ $A = U_0 D V^*$, $U_0, V \in U(n)$.

(SVD decomp.). $D = \mathrm{diag}(s_1, \dots, s_n)$

singular values.

$$A = \underbrace{U_0 V^*}_{U} \underbrace{(V D V^*)}_{|A|} = U \cdot |A|.$$

Proof.

Construction of U:

$$H = \ker(A) \oplus \ker(A)^{\perp}$$

Construction. 1) $U\big|_{\ker(A)} = 0.$

2) $\ker(A)^{\perp} = \ker(|A|)^{\perp} =$

$= \overline{\mathrm{Ran}(|A|^{*})} =$

$= \overline{\mathrm{Ran}(|A|)}.$

Lemma.

$|A| = \sqrt{A^{*}A}$

$|A|^{2} = A^{*}A.$

$\ker(A) = \ker(|A|)$

why:

$\|Ax\|^{2} = \langle Ax, Ax\rangle =$

$= \langle A^{*}Ax, x\rangle = \langle |A|^{2}x, x\rangle$

$= \||A|x\|^{2}.$

i) If $x \in \mathrm{Ran}(|A|)$: $\exists\, v \in H,\ x = |A|\cdot v.$

Set $Ux = Av$

- Need to show such U is well-defined:

if $v' \in H$ s.t. $x = |A|\cdot v'$

$\longrightarrow |A|\cdot v = |A|\cdot v' \dashrightarrow |A|\cdot(v - v') = 0$

$\dashrightarrow v - v' \in \ker(|A|) = \ker(A) \dashrightarrow$

$\longrightarrow A(v - v') = 0 \dashrightarrow Av' = Av.$

- $\|Ux\|^{2} = \|Av\|^{2} = \langle Av, Av\rangle = \langle A^{*}Av, v\rangle =$

$= \langle |A|^{2}v, v\rangle = \||A|v\|^{2} = \|x\|^{2}.$

$\longrightarrow U\big|_{\mathrm{Ran}(|A|)}$ is an isometry.

Then: ii) $x \in \overline{\mathrm{Ran}(|A|)}$ $\dashrightarrow$ U is defined as the unique continuous extension from $\mathrm{Ran}(|A|)$

$$U\Big|_{\overline{\mathrm{Ran}(|A|)}} \longrightarrow \mathrm{Ran}(U) \text{ is } \underline{\text{unitary map}}.$$

Based on the previous proposition, $U : H \longrightarrow H$ is a partial isometry.

<u>Uniqueness</u> .... (take as a homework).

<u>Proposition</u>. Let $A \in B(H)$. The polar decomposition: $A = U_A \cdot |A|$

(1) $U_A^* A = |A|$

(2) $|A^*| = U_A \cdot |A| \cdot U_A^*$

(3) $U_{A^*} = U_A^*$

(4) $A = |A^*| \, U_A$

(5) $A^* = U_A^* A \, U_A^*$, $\quad A = U_A \cdot A^* \, U_A$

<u>Sketch of Proof</u>.

(1) $A = U_A \cdot |A| \dashrightarrow U_A^* A = \underbrace{U_A^* U_A}_{\text{orth. proj. } P_I \text{ onto } \overline{\mathrm{Ran}(|A|)}} \cdot |A| = |A|.$

(2). $U_A \cdot |A| \, U_A^* = \left(U_A |A| U_A^* \right)^* \geq 0.$

$$(U_A|A|U_A^*)^2 = \underbrace{U_A|A|}_{A} U_A^* U_A \underbrace{|A|U_A^*}_{A^*} = A \underbrace{U_A^* \cdot U_A}_{\cancel{|A| = U_A^* A}} A^* =$$

proj. onto $\overline{\mathrm{Ran}(|A|)} = \ker(A)^\perp =$

$= \overline{\mathrm{Ran}(A^*)}$

$\Rightarrow U_A^* U_A \cdot A^* = A^*$

$$= A \cdot A^* = |A^*|^2 \quad \Rightarrow \quad A^* = U_A |A| U_A^*.$$

(3). $A^* = U_{A^*} \cdot |A^*| = \underbrace{U_{A^*} U_A}\, |A| \cdot U_A^*$

$\|$

$|A| U_A^*$ - - $\longrightarrow$ orth. proj. onto $\overline{\mathrm{Ran}(|A|)} = U_A^* \cdot U_A$

$\dashrightarrow U_{A^*} = U_A^*$

(because both are partial isometries with same initial space and final subspace).

...

[Next time: We consider the case $A = A^*$.

$\dashrightarrow U_{A^*} = U_A^* \dashrightarrow U_A^* = U_A$

$\longrightarrow (U_A)^2$ proj. $\dashrightarrow H = H_- \oplus H_0 \oplus H_+$]