Spectral Theorem.

Theorem [Continuous Functional Calculus] Let A be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$. Then there exists a $*$-homomorphism,

$$\Phi_A : C([-\|A\|, \|A\|]) \longrightarrow B(\mathcal{H})$$

(we write $\Phi_A(f) = f(A)$) so that:

(1) $\mathbb{1} : \mathbb{R} \to \mathbb{R}$, $\mathbb{1}(x) = 1 \longrightarrow \Phi_A(\mathbb{1}) = I_{B(\mathcal{H})}$

(2) $\mathcal{X} : \mathbb{R} \to \mathbb{R}$, $\mathcal{X}(x) = x \longrightarrow \Phi_A(\mathcal{X}) = A$

(3) $\|\Phi_A(f)\| \leq \|f\|_\infty$

(4). $*$-homomorphism: (i). $\Phi_A(af + bg) = a\Phi_A(f) + b\Phi_A(g)$ $\forall a, b \in \mathbb{C}$, $f, g \in C([-\|A\|, \|A\|])$

(ii) $\Phi_A(\bar{f}) = (\Phi_A(f))^*$

(iii) $\Phi_A(f \cdot g) = \Phi_A(f) \cdot \Phi_A(g)$

(5) $\sigma(\Phi_A(f)) = f[\sigma(A)] := \{f(\lambda), \lambda \in \sigma(A)\}$.

(6). Φ_A with prop. (1) $\div$ (4) is unique.

Proof:

Construction:

Key Obervations:

(1) If P is a polynomial, then. $\sigma(P(A)) = P[\sigma(A)]$.

(2). If B is a normal operator then $\|B\| = \max_{\lambda \in \sigma(B)} |\lambda|$.

(3). The algebra of polynomials $\mathcal{P}$ in real variable X.

is dense in. $C([-\|A\|, \|A\|])$ w.r.t. Sup-norm (L^∞-norm

How to construct Φ_A:

1) $$\Phi_A(P) := P(A), \text{ for any polynomial } P \in \mathcal{P}$$

$$P = c_0 x^n + c_1 x^{n-1} + \cdots + c_n$$

$$\dashrightarrow \Phi_A(P) = c_0 A^n + c_1 A^{n-1} + \cdots + c_n. \in B(H).$$

Note: $\Phi_A(P)$ is a normal operator; and Φ_A satisfies:

1) $\Phi_A(a \cdot P + b \cdot Q) = a \cdot \Phi_A(P) + b \cdot \Phi_A(Q)$ - linear.

2) $\Phi_A(P \cdot Q) = \cdots = \Phi_A(P) \cdot \Phi_A(Q)$

3) $\Phi_A(\bar{P}) = \bar{c}_0 A^n + \bar{c}_1 A^{n-1} + \cdots + \bar{c}_n \cdot 1_H = (\Phi_A(P))^*$

Thus Φ_A is a $*$-homomorphism between $\mathcal{P}$ and $B(H)$.

2)

2) $\|\Phi_A(P)\|_{B(H)} = \|P(A)\|_{B(H)} = \max_{\lambda \in \sigma(P(A))} |\lambda| = \max_{\lambda \in \sigma(A)} |P(\lambda)| \leq$

$\leq \max_{\lambda \in [-\|A\|, \|A\|]} |P(\lambda)| = \|P\|_\infty .$

3). Admits a unique continuous extension to the closure of $\mathcal{P}$ w.r.t. sup-norm.

By Stone-Weierstrass (key observation #2), $\left(\overline{\mathcal{P}}\right)_{\text{sup-norm}} = C([-\|A\|, \|A\|])$

This shows (1) — (4), (6). [Claim (5): we see later].

Remark

1) $\Phi_A(C[-\|A\|, \|A\|])$ is the smallest C^*-algebra that includes A and 1_H.

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4 If $f_1, f_2 \in C(\mathbb{R})$ s.t. $f_1|_{\sigma(A)} = f_2|_{\sigma(A)}$ then.

$\Phi_A(f_1) = \Phi_A(f_2)$. $\dashrightarrow$ $\|\Phi_A(f)\|_{B(H)} = \|f\|_{L^\infty(\sigma(A))}$.

Cyclic Vectors & Simple Operators.

Assume A is a self-adjoint operator on Hilbert space H.

Definition A vector $\varphi \in H$ is called *cyclic for A* if:

$$\overline{\operatorname{Span}}\{\varphi, A\varphi, A^2\varphi, \dots, A^n\varphi, \dots\} = H.$$

Def. A self-adjoint operator A that admits a cyclic vector is called *simple*.

Theorem [Spectral Measure Form for Simple Operators]. Let A be a simple self-adjoint operator on a Hilbert space H with cyclic vector φ. Then there exists a measure $d\mu_\varphi$ on $[-\|A\|, \|A\|]$ and a unitary map

$$U : H \longrightarrow L^2(\mathbb{R}, d\mu_\varphi)$$

such that:

(1) $(U\varphi)(x) = 1, \ \forall x \in \mathbb{R}$. : $\varphi \xmapsto{U} \mathbb{1}$

(2) $(UA\varphi)(x) = x, \ \forall x$: $A\varphi \mapsto x$

(3) $(UA\psi)(x) = x \cdot (U\psi)(x), \ \forall x \in \mathbb{R}, \forall \psi \in H.$

$$\begin{array}{ccc} H & \xrightarrow{U} & L^2(\mathbb{R}, d\mu_\varphi) \\ A \downarrow & & \downarrow M_x \\ H & \xrightarrow{U} & L^2(\mathbb{R}, d\mu_\varphi) \end{array}$$

$M_x : L^2(\mathbb{R}, d\mu_\varphi) \to L^2(\mathbb{R}, d\mu$

$(M_x f)(x) = x \cdot f(x$

Further: $\mu_\varphi(\mathbb{R}) = \mu_\varphi([-\|A\|, \|A\|]) = \|\varphi\|^2.$

Proof. ① Construction of measure:

Fix φ. cyclic vector. Consider:

$$\forall f \in C([-\|A\|, \|A\|]) \dashrightarrow \ell_\varphi(f) = \langle f(A)\varphi, \varphi \rangle.$$

where $f(A) = \Phi_A(f)$ has been constructed by the Continuous Functional Calculus.

Note: $|\ell_\varphi(f)| = |\langle f(A)\varphi, \varphi\rangle| \le \|f(A)\| \cdot \|\varphi\|^2 \le \|f\|_\infty \cdot \|\varphi\|^2$

$\rightarrow$ ℓ_φ is a bounded linear functional on $C([-\|A\|, \|A\|])$

by Riesz-Markov Theorem (i.e. the duality theorem for $C([-\|A\|, \|A\|])$

$\exists!$ (signed) measure μ_φ on the Borel σ-algebra over $[-\|A\|, \|A\|]$

$$\text{s.t.} \quad \ell_\varphi(f) = \int_I f \, d\mu_\varphi.$$

$I = [-\|A\|, \|A\|]$.

Claim: ℓ_φ is a positive functional s.t. μ_φ is a measure. (not only signed measure)

$\Updownarrow$ Definition:

If $f \in C([-\|A\|, \|A\|])$ s.t. $f(x) \ge 0$, $\forall x$

Then $\ell_\varphi(f) \ge 0$.

why: Take $f \in C(I)$, $f(x) \ge 0$. Construct $g(x) = \sqrt{f(x)}$.

Note $g \in C(I)$. and $f = g^2$

$$\Rightarrow \ell_\varphi(f) = \langle f(A)\varphi, \varphi\rangle = \langle g(A) \cdot g(A)\varphi, \varphi \rangle = \|g(A)\varphi\|^2 \ge 0.$$

② <u>Construction of unitary U.</u>

$$\mu_\varphi(\mathbb{R}) = \underbrace{\int_I \mathbb{1}\, d\mu_\varphi}_{} = \ell_\varphi(\mathbb{1}) = \langle \underbrace{\Phi_A(\mathbb{1})}_{1_H}\varphi,\varphi\rangle = \langle\varphi,\varphi\rangle = \underbrace{\|\varphi\|^2}_{}.$$

$$\|\mathbb{1}\|^2_{L^2([-\|A\|,\|A\|],d\mu_\varphi)} = \|\varphi\|^2$$

let $n, m \geq 0$ two integers. $A^{n+m} = A^n \cdot A^m$

$$\langle x^n, x^m\rangle_{L^2([-\|A\|,\|A\|],d\mu_\varphi)} = \int_I x^{n+m}\, d\mu_\varphi = \ell_\varphi(x^{n+m}) =$$

$$= \langle \Phi_A(x^{n+m})\varphi,\varphi\rangle_H = \langle A^{n+m}\varphi,\varphi\rangle = \langle A^n\varphi, A^m\varphi\rangle.$$

~~In particular~~ By linearity:

$$\langle P(x), P(x)\rangle = \langle P(A)\varphi, P(A)\varphi\rangle.$$

$$\|P\|^2_{L^2(I,d\mu_\varphi)} = \|P(A)\varphi\|^2_H$$

We obtained an <u>isometry</u>: $V : \mathcal{P} \longrightarrow \mathrm{span}\{P(A)\varphi,\ P\in\mathcal{P}\}$.

$\uparrow$ $L^2(I,d\mu_\varphi)$-norm $\qquad\qquad$ $\uparrow$ H-norm

We extend $\overset{\text{continuously}}{\text{the}}$ isometry V to a unique onto isometry. (7)

$$V : L^2(I, d\mu_\varphi) \longrightarrow \overline{\operatorname{span}}\{\varphi, A\varphi, \dots\} = H.$$

$\Rightarrow$ V is unitary. $\longrightarrow$ let $U = V^{-1} : H \longrightarrow L^2(I, d\mu_\varphi)$.

<u>Note:</u> $U(P(A)\varphi) = P(x)$.

$$\dashrightarrow \quad U(\varphi) = 1$$

$$U(A\varphi) = x$$

and.

If $\psi \in H$. Fix $\varepsilon > 0$ $\exists\, P \in \mathcal{P}$ s.t. $\|\psi - P(A)\varphi\| < \varepsilon$.

$$\|A\psi - A \cdot P(A)\varphi\| < \varepsilon \cdot \|A\|$$

$$\|U(A\psi) - x \cdot P(x)\|_{L^2(I, d\mu_\varphi)} = \|A\psi - A\,P(A)\varphi\| < \varepsilon \cdot \|A\|$$

$$\|U(\psi) - P(x)\|_{L^2(I, d\mu_\varphi)} = \|\psi - P(A)\varphi\| < \varepsilon.$$

$$\|U(A\psi) - x \cdot U(\psi)\|_{L^2} \le \|U(A\psi) - x \cdot P(x)\|_{L^2} +$$

$$+ \|x \cdot (U(\psi) - P(x))\|_{L^2(I, d\mu_\varphi)} \le \varepsilon \cdot \|A\| + \|x\|_{L^\infty(I)} \cdot \varepsilon = 2\varepsilon \cdot \|A\|$$

Since ε is arbitrary $\Rightarrow$ $U(A\psi) = x \cdot U(\psi)$.

Extension to general self-adjoint operators:

Lemma 1. Assume $A \in B(H)$. Let $\varphi \in H$ and

$$V = \overline{\text{Span}} \{ \varphi, A_1 A_2 \dots A_n \varphi : \underset{n=1,2,\dots}{A_1, \dots, A_n \in \{A, A^*\}} \}.$$

Let $\psi \in V^\perp$ and denote $W = \overline{\text{Span}} \{ \psi, A_1 A_2 \dots A_n \psi ; A_1 \dots A_n \in \{A, A^*\} \}$

Then $V \perp W$.

Lemma 2 Assume $A \in B(H)$. Then there exists an orthonormal set of H.

$\{ \varphi_1, \varphi_2, \dots, \varphi_N \}$ (where N can be infinite). such that:

$$H_{\varphi_k} = \overline{\text{Span}} \{ \varphi_k, A_1 A_2 \dots A_n \varphi_k ; \underset{n=1,2,\dots}{A_1 \dots A_n \in \{A, A^*\}} \}.$$

perform an orthogonal decomposition of H:

$$H = \bigoplus_{k \geq 1}^{N} H_{\varphi_k}$$

Consequence: ~~If A = A*~~ If $A = A^*$ then

$$H = \bigoplus_{k=1}^{N} H_{\varphi_k}$$

and each H_{φ_k} is invariant to the action of A, and $A|_{H_{\varphi_k}}$ is simple, having φ_k as cyclic vector.

<u>Theorem</u> [Multiplication Operator Form] Let A be a bounded self-adjoint operator on a Hilbert space H. Then there exist measures $\{\mu_j\}_{j=1}^{N}$ (N finite or infinite) on $[-\|A\|, \|A\|]$ and a unitary map

$$U : H \longrightarrow \bigoplus_{j=1}^{N} L^2(\mathbb{R}, d\mu_j)$$

so that $U A U^{-1}$ is multiplication by x, i.e.,

$$\left(UA\varphi\right)_j(x) = x \cdot \left(U\varphi\right)_j(x), \quad \forall x \in \mathbb{R}, \ \forall \varphi \in H.$$

<u>Proof</u>: Put together lemma 2 & Spectral measure in the cyclic case.