L29



# Spectral Theorems (2)

<u>Theorem</u> [Multiplicity Theorem]. Let $A$ be a bounded self-adjoint operator on $H$. Then there exist measures $\{\mu_\ell\}_{\ell\geq 1}$, $\mu_\infty$ supported in $[-\|A\|, \|A\|]$ that are mutually singular ($\mu_j \perp \mu_\ell$, for each integer $j, \ell$ including $\infty$), and a unitary operator

$$U : H \longrightarrow \bigoplus_{\ell=1}^{\infty} L^2(\mathbb{R}, d\mu_\ell; \mathbb{C}^\ell) \oplus L^2(\mathbb{R}, d\mu_\infty; \ell^2).$$

such that

$$(UA\varphi)_\ell(x) = x \cdot (U\varphi)_\ell(x), \quad \forall x, \forall \ell \geq 1.$$

<u>Note</u>: $(U\varphi)_\ell \in \mathbb{C}^\ell$.

---

<u>Proof</u>: Result follows the Multiplication Operator Form by collecting together. $x$-components from multiple $\mu_j$'s from the Multiplication Operator form into one $\ell$-long vector.

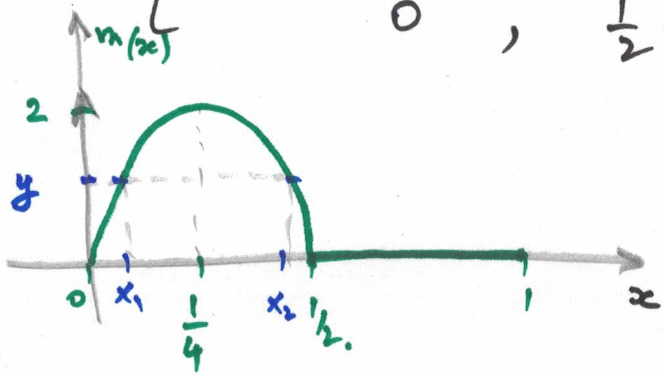
—.

**Example.** Consider $A : L^2[0,1] \to L^2[0,1]$

$$Af = m \cdot f,$$

where
$$m(x) = \begin{cases} 2\sin(2\pi x), & 0 \le x \le \frac{1}{2} \\ 0, & \frac{1}{2} < x \le 1. \end{cases}$$

$$(Af)(x) = \begin{cases} 2\sin(2\pi x) f(x), & 0 \le x \le \frac{1}{2} \\ 0, & \frac{1}{2} < x \le 1. \end{cases} = m(x) \cdot f(x)$$




We know by now: $\sigma(A) = \mathrm{Ran}(m) = \{m(x), x \in [0,1)\} = [0,2]$

$$r(A) = \|A\| = 2.$$

Let $y = m(x)$, Assume $y \in [0,2]$

$$y = 2\sin(2\pi x) \dashrightarrow x_1 = \frac{1}{2\pi}\arcsin\left(\frac{y}{2}\right)$$

$$x_2 = \frac{1}{2} - \frac{1}{2\pi}\arcsin\left(\frac{y}{2}\right).$$

We seek a unitary map

$$f \in L^2[0,1] \xrightarrow{U} \Phi = U(f) \in L^2\left[[0,2], d\mu; \mathbb{C}^2\right] \oplus \ell^2$$

$$\ell^2 \simeq L^2\left[\{0\}, \delta; \ell^2(\mathbb{Z})\right]$$

We construct $U$ as follows:

$$f \longmapsto \Phi = \varphi \oplus \varphi_0 , \qquad \varphi : [0,2] \to \mathbb{C}^2$$
$$\varphi_0 \in \ell^2$$

$$\varphi(y) = \begin{bmatrix} f(x_1) \\ f(x_2) \end{bmatrix} = \begin{bmatrix} f\left(\frac{1}{2\pi} \arcsin\left(\frac{y}{2}\right)\right) \\ f\left(\frac{1}{2} - \frac{1}{2\pi}\arcsin\left(\frac{y}{2}\right)\right) \end{bmatrix}.$$

$\varphi_0 \in \ell^2(\mathbb{Z})$, $\quad (\varphi_0)_n = \langle f|_{[\frac{1}{2},1]}, e_n \rangle$. , where $\{e_n\}_{n\in\mathbb{Z}}$ ONB for $L^2[\frac{1}{2},1]$.

For instance: $e_n(x) = \sqrt{2}\, e^{4\pi i n x}$

Need to find the measure $\mu$ s.t. $U$ is unitary.

$$\|\Phi\|^2_{L^2([0,2],d\mu;\mathbb{C}^2)\oplus\ell^2} = \|\varphi\|^2_{L^2([0,2],d\mu)} + \|\varphi_0\|^2_{\ell^2}$$

$$\|\varphi_0\|^2_{\ell^2} = \sum_{n=-\infty}^{\infty} |(\varphi_0)_n|^2 = \sum_{n\in\mathbb{Z}} |\langle f|_{[\frac{1}{2},1]}, e_n\rangle|^2 = \|f|_{[\frac{1}{2},1]}\|_2^2 = \int_{\frac{1}{2}}^{1} |f(x)|^2 dx$$

$$\|\varphi\|^2_{L^2([0,2],d\mu)} = \int_0^2 \|\varphi(y)\|^2 d\mu = \int_0^2 \left[ \left|f\left(\tfrac{1}{2\pi}\arcsin\left(\tfrac{y}{2}\right)\right)\right|^2 + \left|f\left(\tfrac{1}{2} - \tfrac{1}{2\pi}\arcsin\left(\tfrac{y}{2}\right)\right)\right|^2 \right] d\mu(y)$$

$$x = \frac{1}{2\pi}\arcsin\left(\frac{y}{2}\right). \quad : y \in [0,2) \longrightarrow x \in [0, \tfrac{1}{4}].$$

$$dx = \frac{1}{2\pi} \frac{1}{\sqrt{1-\frac{y^2}{4}}} \cdot \frac{1}{2}\, dy = \frac{1}{2\pi} \frac{1}{\sqrt{4-y^2}}\, dy.$$

<u>Need to choose</u>: $d\mu(y) = \frac{1}{2\pi}\frac{1}{\sqrt{4-y^2}}\, dy \overset{\text{becomes}}{=} dx.$

$$\|\varphi\|^2_{L^2([0,2],d\mu)} = \int_0^{\frac{1}{4}} \left[ |f(x)|^2 + |f(\tfrac{1}{2}-x)|^2 \right] dx = \int_0^{\frac{1}{2}} |f(x)|^2 dx$$

$$\Rightarrow \|\Phi\|^2_{L^2([0,2],d\mu;\mathbb{C}^2)\oplus \ell^2} = \int_0^{1/2} |f(x)|^2 dx + \int_{1/2}^{1} |f(x)|^2 dx = \int_0^1 |f(x)|^2 dx = \|f\|_2^2$$

---

Action of $\mathbf{A}$:

$$UAf = \psi \oplus \psi_0$$

$$\psi_0 = 0,$$

$$\psi(y) = \begin{bmatrix} (Af)(\frac{1}{2\pi}\arcsin(\frac{y}{2})) \\ (Af)(\frac{1}{2} - \frac{1}{2\pi}\arcsin(\frac{y}{2})) \end{bmatrix} = \begin{bmatrix} \overbrace{2\cdot\sin(2\pi\frac{1}{2\pi}\arcsin(\frac{y}{2}))}^{y.}\cdot f(\frac{1}{2\pi}\arcsin(\frac{y}{2})) \\ \underbrace{2\cdot\sin(2\pi\frac{1}{2} - 2\pi\frac{1}{2\pi}\arcsin(\frac{y}{2}))}_{y}\cdot f(\dots) \end{bmatrix}$$

$$= y\cdot \begin{bmatrix} f(\frac{1}{2\pi}\arcsin(\frac{y}{2})) \\ f(\frac{1}{2} - \frac{1}{2\pi}\arcsin(\frac{y}{2})) \end{bmatrix} = y\cdot \varphi(y)$$

where $Uf = \varphi \oplus \varphi_0$

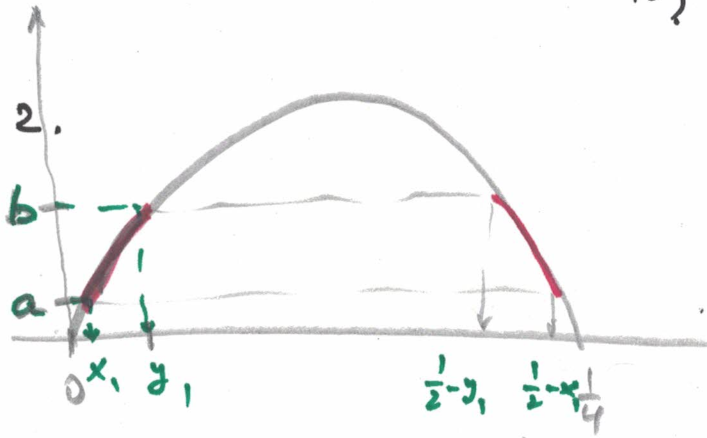
$$(UAf)(y) = (y\cdot\varphi(y)) \oplus (0\cdot\varphi_0) \underset{\substack{\uparrow \\ \text{w.r.t. } \underline{\underline{\mu\oplus\delta}}\text{-a.e.}}}{=} y\cdot(\varphi\oplus\varphi_0)$$

## Spectral Projectors:

Let $\Omega = (a,b)$, $\quad 0 \leq a < b \leq 2$.

Let $x_1 = \frac{1}{2\pi} \arcsin\left(\frac{a}{2}\right)$

$y_1 = \frac{1}{2\pi} \arcsin\left(\frac{b}{2}\right)$.




$$\Omega \longrightarrow P_\Omega \cdot f = \mathbb{1}_{(x_1,y_1) \cup (\frac{1}{2}-y_1, \frac{1}{2}-x_1)} \cdot f$$

<u>Claim</u>: $P_\Omega$ are orthogonal projections.

$$(P_\Omega \cdot f)(x) = \left(U^{-1}\, \mathbb{1}_\Omega \cdot U f\right)(x). = \begin{cases} f(x), & x \in (x_1,y_1) \cup (\frac{1}{2}-y_1, \frac{1}{2} \\ 0, & \text{otherwise.} \end{cases}$$

---

<u>Claim</u>: Follows from Borel Functional Calculus:

$$P_\Omega = \mathbb{1}_\Omega(A).$$

**Theorem** [Borel Functional Calculus] Let $A$ be a bounded self-adjoint operator on a Hilbert space $\mathcal{H}$, and let $\mathcal{B}([-\|A\|,\|A\|])$ denote the bounded Borel functions on $[-\|A\|,\|A\|]$.

Then the $*$-homomorphism $\Phi_A$ constructed in the continuous functional calculus extends to $\Phi_A : \mathcal{B}([-\|A\|,\|A\|]) \longrightarrow B(\mathcal{H})$ and satisfies:

1) $\Phi_A$ is $*$-homomorphism:

$$\Phi_A(af+bg) = a\Phi_A(f) + b\Phi_A(g)$$
$$\Phi_A(\bar{f}) = (\Phi_A(f))^*$$
$$\Phi_A(f\cdot g) = \Phi_A(f)\cdot\Phi_A(g).$$

2) $\|\Phi_A(f)\| \leq \|f\|_\infty$, $\forall f \in \mathcal{B}([-\|A\|,\|A\|])$

3) If $(f_n)_{n\geq 1}$, $f$ are in $\mathcal{B}([-\|A\|,\|A\|])$ such that:

i) $\sup_{x,n} |f_n(x)| < \infty$, (i.e. $\sup_n \|f_n\|_\infty < \infty$).

ii) $\lim_{n\to\infty} f_n(x) = f(x)$, for every $x$

Then $\underset{n\to\infty}{s-\lim}\, \Phi_A(f_n) = \Phi_A(f)$.

strong limit. i.e. $\forall \varphi \in \mathcal{H}$, $\lim_{n\to\infty} \|(\Phi_A(f_n) - \Phi_A(f))\varphi\| = 0$.

<u>Proof</u>. Multiplicative Theorem: Fix $f \in \mathcal{B}([-\|A\|, \|A\|])$.

$$\begin{tikzcd}
H \arrow[r, "U"] \arrow[d, "\Phi_A(f)"'] & \bigoplus L^2(\mathbb{R}, d\mu_\ell; \mathbb{C}^\ell) \oplus L^2(\mathbb{R}, d\mu_\infty; \ell^2) \arrow[d, "f\cdot \,\equiv M_f"] \\
H & \bigoplus L^2(\mathbb{R}, d\mu_\ell; \mathbb{C}^\ell) \oplus L^2(\mathbb{R}, d\mu_\infty; \ell^2). \arrow[l, "U^{-1}"']
\end{tikzcd}$$

$$M_f : L^2(\ldots) \oplus L^2(\ldots) \longrightarrow L^2(\ldots) \oplus L^2(\ldots)$$

$$M_f(g) = f \cdot g \qquad \text{(multiplicative)}.$$

$$\underline{\Phi_A(f) := U^{-1} M_f U}$$

$$\begin{cases} \text{If } f = \mathbb{1} \longrightarrow \Phi_A(\mathbb{1}) = I_H \\ f = x \dashrightarrow \Phi_A(x) = A. \end{cases}$$

③: Consequence of Lebesgue Dominated Convergence Theorem:

take $\eta \in H$

$$\lim_{n\to\infty} \|(\Phi_A(f_n) - \Phi_A(f))\eta\|^2 = \|(M_{f_n} - M_f) g\|^2_{L^2(\ldots)} =$$

$$g = U(\eta).$$

$$= \lim_{n\to\infty} \sum_\ell \int \underbrace{|f_n(x) - f(x)|^2}_{\leq 4M^2} \cdot \underbrace{\|g(x)\|^2}_{\in L^1} d\mu_\ell + \ldots \overset{DCT}{=} 0.$$

$$M = \sup_n (|f_n(x)|, |f(x)|).$$

Corollary of Borel Calculus:

1. $\sigma(\Phi_A(f)) = f[\sigma(A)]$.

Resolution of identity:

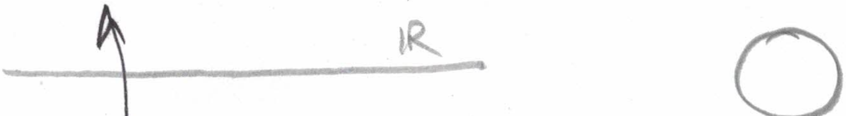
2. $E_t = \Phi_A\left(1_{(-\infty,t]}\right)$, $\forall t$.

.... $\Rightarrow$ All results extend to normal operators!

[ ---> Unbounded operators ... ].

[ $A \dashrightarrow x \mapsto f(x) = \frac{1+ix}{1-ix} = -\frac{x-i}{x+i}$ : $\left|\frac{x-i}{x+i}\right| = 1$.

$\langle A\varphi,\psi\rangle = \langle\varphi, A\psi\rangle$.




A self-adj.: $A^* = A$

1. $D(A^*) = D(A)$.
2. $\forall \varphi,\psi \in D(A)$: $\langle A\varphi,\psi\rangle = \langle\varphi, A\psi\rangle$.

$U = (A - i\cdot 1)(A + i\cdot 1)^{-1} \xrightarrow[\text{Theorem}]{\text{Spectral}}$