(L2) (1)

Metric Spaces - Cont.

$(\overline{X}, d)$ metric space.

Definition The metric space $(\overline{X}, d)$ is said separable if there is a countable dense subset $A \subset \overline{X}$.

Examples:

All $L^p([0,1])$, $0<p<\infty$. $p \neq \infty$.

are separable.

$L^p(\mathbb{R})$, $0<p<\infty$ $\rightarrow$ are separable.

$\bigcup_{n\geq 0} \left\{ \sum_{k=0}^{n} c_k x^k , \quad c_0, c_1 \dots c_n \in \mathbb{Q} \right\}$ $\rightarrow$ use Weierstrass approximation.

$L^\infty([0,1])$ is not separable. $\leftarrow$

$\left\{ \mathbb{1}_{[0,t]} , \quad t \in [0,1] \right\}$. uncountable set.

$$d(\mathbb{1}_{[0,t]}, \mathbb{1}_{[0,s]}) = \begin{cases} 1, & t \neq s \\ 0, & t = s. \end{cases}$$

$C([0,1])$ is separable. $\leftarrow$ by Weierstrass approximation theorem.

$$C[0,1] \subset L^{\infty}([0,1]) \subset L^{p}([0,1]) \subset L^{1}([0,1]) \subset L^{q}([0,1]).$$

$1<p<\infty$ $\qquad$ $0<q<1.$

complete w.r.t. the distance: $\quad d_\infty \quad d_\infty \quad d_p \quad d_1 \quad d_q.$

$(C[0,1], d_\infty)$ is <u>not</u> dense in $(L^\infty[0,1], d_\infty)$

$C[0,1]$ is dense in $(L^p[0,1], d_p)$, $0<p<\infty$

$L^\infty[0,1]$ is dense in $(L^p[0,1], d_p)$, $0<p<\infty$.

<u>Compactness in Metric Spaces</u>:

Let (X,d) be a metric space and $A \subset X$.

<u>Definitions.</u>

(1) The set A is said <u>bounded</u> if $\exists R>0, \exists x \in X$ s.t. $A \subset B_R(x)$.

(2) The set A is said <u>totally bounded</u>:

$$\left[\forall r>0 \ \exists N=N(r), \exists x_1, \dots x_N \in X \text{ s.t. } A \subset \bigcup_{k=1}^{N} B_r(x_k) \right.$$

(3) The set A is said sequentially compact if.

any sequence $(x_n)_{n\geq 1}$ in A admits a convergent subsequence in A

$\exists (n_k)_{k\geq 1}$ s.t. $(x_{n_k})_{k\geq 1}$ is convergent in $\underline{A}$.

<u>Theorem</u> [MATH 631 $\to$ Royden - Fitzpatrick]. Let $(\mathbb{X}, d)$ be a metric space and $A \subset \mathbb{X}$ a subset. The following are equivalent:

(1) A is complete and totally bounded.

(2) A is compact

(3) A is sequentially compact.

$\to$ Arzela - Ascoli Theorem. $\dashrightarrow$ equicontinuous. $\to$ uniformly bounded.

— compactness in L^p, $C[0,1]$.

Normed Vector Spaces

Let $(V,+)_{\mathbb{C}}$ be a complex vector space.

Definition. A map $\|\cdot\| : V \longrightarrow \mathbb{R}$ is called norm if:

(1) (positivity). $\|x\| \geq 0$, $\forall x \in V$. (i)

$\|x\| = 0$ iff $x = 0$. (ii)

(2) (homogeneity) $\|a \cdot x\| = |a| \cdot \|x\|$, $\forall a \in \mathbb{C}$, $\forall x \in V$

(3) (triangle inequality) $\|x+y\| \leq \|x\| + \|y\|$, $\forall x,y \in V$.

Def. If the map $\|\cdot\|$ satisfies (i), $\|0\| = 0$, (2) & (3) then it is called semi-norm.

Remark.

If. $\|\cdot\|$ is a semi-norm on V, then:

1) $S = \{x \in V : \|x\| = 0\}$. ---> set of null-vectors.

is a linear space, subspace of V.

2) $|||\cdot||| : V/S \longrightarrow \mathbb{R}$, $|||\hat{x}||| = \|y\|$, $\forall y \in \hat{x}$

↳ is a norm on V/S.

$\longrightarrow$

Def. The pair $(V, \|\cdot\|)$ of a vector space with a norm on V is called a NORMED VECTOR SPACE. (normed linear space).

Examples. For $1 \le p \le \infty$.

$$\left(L^p, \|\cdot\|_p\right), \quad \text{where } \|f\|_p = d_p(f, 0)$$

$L^p(X, d\mu)$: $f \in L^p$:

$$\|f\|_p = \begin{cases} \left(\int_X |f(x)|^p \, d\mu(x)\right)^{1/p}, & 1 \le p < \infty \\ \operatorname{ess\,sup}_{x \in X} |f(x)|, & p = \infty. \end{cases}$$

If. $(V, \|\cdot\|)$ is a normed vector space, then the induced distance $d : V \times V \to \mathbb{R}$ is defined by:

$$d(x,y) = \|x - y\|$$

and (V, d) is a metric space.

$(V, \|\cdot\|)$ is said complete if. (V, d) is a complete metric space.

Definition. A complete normed vector space $(V, \|\cdot\|)$ is called a Banach space.

Vector Spaces with Scalar Product

Let $(V,+)_{\mathbb{C}}$ be a complex vector space.

Definition The map $\langle\cdot,\cdot\rangle : V\times V \to \mathbb{C}$ is called a scalar product (or inner product) on V if:

(1) (positivity) $\forall x\in V$, $\langle x,x\rangle \geq 0$. (i)

$\langle x,x\rangle = 0$ iff $x=0$ (ii).

complex conjugate

(2). (skew-symmetry): $\forall x,y\in V$, $\langle x,y\rangle = \overline{\langle y,x\rangle}$. $\overline{x+iy} = x-iy$

(3) (linearity): $\forall x,y,z\in V$, $\forall a,b\in\mathbb{C}$,

$$\langle ax+by, z\rangle = a\langle x,z\rangle + b\langle y,z\rangle.$$

~~Remark~~ Def. The pair $(V, \langle\cdot,\cdot\rangle)$ is called a vector space with scalar product.

Remark. If $\langle\cdot,\cdot\rangle$ is a scalar product on V, then

$$\|x\| = \sqrt{\langle x,x\rangle}$$

defines a norm on V, called the induced (associated) norm.

Example.

$V = L^2([0,1])$,

$f, g \in L^2 \longrightarrow \langle f, g\rangle = \int_0^1 f(x)\cdot\overline{g(x)}\, dx$

$(L^2, \langle\cdot,\cdot\rangle)$ is a vector space with scalar product.

Proposition [Jordan – Neumann]. ~~Assume~~ Let V be a vector space.

(1) Assume $(V, \langle\cdot,\cdot\rangle)$ is a vector space with scalar product. let $\|x\| = \sqrt{\langle x, x\rangle}$ denote the induced norm. Then $\|\cdot\|$ satisfies:

$$\forall x, y \in V, \qquad \|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

(parallelogram relation).

(2). Assume $(V, |||\cdot|||)$ is a normed vector space that satisfies:

$$\forall x, y \in V, \quad |||x+y|||^2 + |||x-y|||^2 = 2\left(|||x|||^2 + |||y|||^2\right).$$

Then there exists a (unique) scalar product $\langle\langle\cdot,\cdot\rangle\rangle$ on V such that $|||x||| = \sqrt{\langle\langle x, x\rangle\rangle}$.

Why:

(1) : direct computation:

$$\|x+y\|^2 + \|x-y\|^2 = \langle x+y, x+y\rangle + \langle x-y, x-y\rangle =$$

$$= \langle x,x\rangle + \langle y,y\rangle + \cancel{\langle x,y\rangle} + \cancel{\langle y,x\rangle} +$$

$$+ \langle x,x\rangle + \langle y,y\rangle - \cancel{\langle x,y\rangle} - \cancel{\langle y,x\rangle}. = 2\langle x,x\rangle + 2\langle y,y\rangle.$$

$$= 2\|x\|^2 + 2\|y\|^2.$$

(2). Polarization identity

Sketch: consider the real vector space case.

$$|||x+y|||^2 = |||x|||^2 + |||y|||^2 + 2\langle\langle x,y\rangle\rangle.$$

$$|||x-y|||^2 = |||x|||^2 + |||y|||^2 - 2\langle\langle x,y\rangle\rangle.$$

$$\boxed{\langle\langle x,y\rangle\rangle = \frac{1}{2}\left[|||x+y|||^2 - |||x-y|||^2\right].}$$

In the complex case

Define:

$$\langle\langle x,y\rangle\rangle = \frac{1}{4}\left[|||x+y|||^2 + i\,|||x+iy|||^2 - |||x-y|||^2 - i\,|||x-iy|||\right.$$

Polarization Identity

Prove → $\langle\langle \cdot,\cdot\rangle\rangle$ is a scalar product.

--> Homework.

Proposition (Cauchy–Schwartz Inequality).

Assume $(V, \langle\cdot,\cdot\rangle)$ is a vector space with a scalar product.

Then:

$$\forall x, y \in V, \qquad |\langle x,y\rangle| \leq \sqrt{\langle x,x\rangle} \cdot \sqrt{\langle y,y\rangle}.$$

Definition. If $(V, \langle\cdot,\cdot\rangle)$ is a complete vector space with scalar product then it is called a Hilbert space.

complete → w.r.t. induced norm & distance

$$(V, \langle\cdot,\cdot\rangle) \longrightarrow \|x\| = \sqrt{\langle x,x\rangle} \longrightarrow d(x,y) = \|x-y\| = \sqrt{\langle x-y, x-y\rangle}.$$

Example

$$(L^2[0,1], \langle\cdot,\cdot\rangle) \text{ is a Hilbert space.}$$

CAVEAT: We shall assume all Hilbert spaces that we deal with are SEPARABLE.