

Convexity and Duality

Theorem [Hahn-Banach Theorem - The Real Case].

Let \underline{X} be a real vector space and $p: \underline{X} \rightarrow \mathbb{R}$ a convex function. Let $\underline{Y} \subset \underline{X}$ be a linear subspace and $l: \underline{Y} \rightarrow \mathbb{R}$ a linear map obeying:

$$l(y) \leq p(y), \quad \forall y \in \underline{Y}$$

Then there exists a linear map $L: \underline{X} \rightarrow \mathbb{R}$ such that:

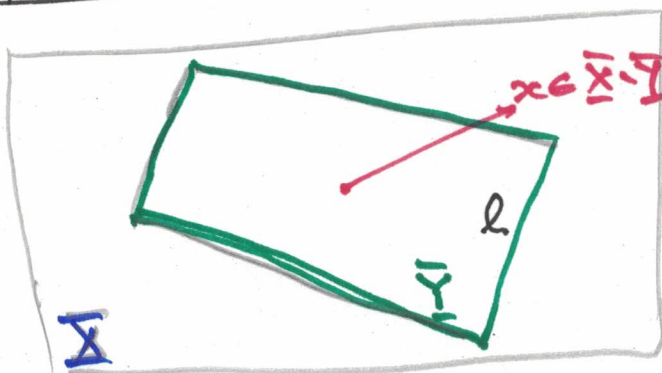
$$(1) \quad L(x) \leq p(x), \quad \forall x \in \underline{X}$$

$$(2) \quad L|_{\underline{Y}} = l.$$

Recall: p is convex: $p((1-t)x + ty) \leq (1-t)p(x) + tp(y)$
 $\forall x, y \in \underline{X}, \forall t \in [0, 1]$.

(Typically: $p(x) = a \cdot \|x\| \dots$).

Idea of Proof:



Step 1 (Main): Extension by 1 dim

$$L \rightarrow \tilde{L} : \text{span}\{Y \oplus x\} \rightarrow \mathbb{R}$$

...

Step 2:

* [If X is normed v.s. & separable]

* ----> extension in a countable # of steps

* [Otherwise]

→ Use Zorn's lemma.

to obtain an extension to \bar{X}

Relationship to ONB:

Assume $E = \{e_\alpha : \alpha \in A\}$ an orthonormal set in H

↳ Hilbert space.

$$1) \quad \bar{Y} = \overline{\text{span}\{e_\alpha\}}$$

either $\bar{Y} = H \rightarrow$ done.

or $\bar{Y} \subsetneq H \rightarrow \exists e' \in H \setminus \bar{Y}$

$\|e'\| = 1 \times e' \perp \{e_\alpha\}_{\alpha \in A}$

$$\rightarrow E' = \{e_\alpha, \alpha \in A\} \cup \{e'\}$$

2) Repeat



Proof of Hahn-Banach (Real Case)

(3)

On Y : $l(y) \leq p(y)$, $\forall y \in Y$ $y \rightarrow -y \rightarrow -l(y) \leq p(-y)$.

$$\Rightarrow 0 \leq p(y) + p(-y) \quad \text{and} \quad 0 \leq p(0).$$

Step 1: Extension.

Lemma Let $Y \subset \underline{X}$ be a ^{proper} linear subspace of the real vector space \underline{X} .

Let $p: \underline{X} \rightarrow \mathbb{R}$ be convex and $l: Y \rightarrow \mathbb{R}$ be a linear map such that $l(y) \leq p(y)$, $\forall y \in Y$. Let $x \in \underline{X} \setminus Y$, $x \neq 0$.

and set $\tilde{Y} = Y \oplus \text{span}\{x\} = \{y + \alpha \cdot x; y \in Y, \alpha \in \mathbb{R}\} \subset \underline{X}$.

Then there exists $\tilde{L}: \tilde{Y} \rightarrow \mathbb{R}$ ^{a linear map} such that:

$$(1) \quad \tilde{L}(z) = p(z), \quad \forall z \in \tilde{Y}$$

$$(2) \quad \tilde{L}|_Y = l.$$

Proof of Lemma.

Since $\forall z \in \tilde{Y}$, $z = y + \alpha \cdot x$, $y \in Y$, $\alpha \in \mathbb{R}$

\rightarrow any linear map \tilde{L} on \tilde{Y} should satisfy:

$$\tilde{L}(z) = \tilde{L}(y + \alpha \cdot x) = \underbrace{\tilde{L}(y)} + \alpha \cdot \tilde{L}(x)$$

$$\text{We want } \tilde{L}|_Y = l \Rightarrow l(y)$$

$$\rightarrow \tilde{L}(z) = l(y) + \alpha \cdot \tilde{L}(x).$$

We need: Find a $\lambda \in \mathbb{R}$ ($\lambda \rightarrow \tilde{L}(x)$).

such that $\tilde{L}(y + \alpha \cdot x) = l(y) + \alpha \cdot \lambda$

satisfies: $\tilde{L}(y + \alpha \cdot x) \leq p(y + \alpha \cdot x), \forall y \in \bar{Y}$
 $\forall \alpha \in \mathbb{R}.$

\Leftrightarrow

$$l(y) + \alpha \cdot \lambda \leq p(y + \alpha \cdot x), \forall y \in \bar{Y}$$
$$\forall \alpha \in \mathbb{R}.$$

If $\alpha = 0 \rightarrow l(y) \leq p(y), \forall y \in \bar{Y}$ ok by hypothesis.

If $\alpha > 0$: $l(y) + \alpha \cdot \lambda \leq p(y + \alpha \cdot x).$
 $\lambda \leq \frac{p(y + \alpha \cdot x) - l(y)}{\alpha}$: need.

If $\alpha < 0 \rightarrow$ let $\beta = -\alpha > 0$:

$$l(y) - \beta \cdot \lambda \leq p(y - \beta \cdot x).$$

$$\lambda \geq \frac{l(y) - p(y - \beta \cdot x)}{\beta}$$

Given $(\alpha, \beta): \alpha, \beta > 0 \rightarrow$

$$\lambda \in \left[\frac{l(y) - p(y - \beta \cdot x)}{\beta}, \frac{p(y + \alpha \cdot x) - l(y)}{\alpha} \right]$$

$I_{y; \alpha, \beta}.$

Claim: For any finite $\{(\alpha_1, \beta_1), \dots, (\alpha_N, \beta_N)\}$:

$$\bigcap_{k=1}^N I_{y; \alpha_k, \beta_k} \neq \emptyset.$$

$$\Leftrightarrow \forall \alpha, \beta > 0 \quad \forall y, \tilde{y} \in Y \quad \frac{l(y) - p(y - \beta x)}{\beta} \leq \frac{p(\tilde{y} + \alpha x) - l(\tilde{y})}{\alpha} \quad (*)$$

Need to check (*):

$$(*) \Leftrightarrow \alpha l(y) - \alpha p(y - \beta x) \leq \beta p(\tilde{y} + \alpha x) - \beta l(\tilde{y}).$$

$$\Leftrightarrow \alpha l(y) + \beta l(\tilde{y}) \leq \alpha p(y - \beta x) + \beta p(\tilde{y} + \alpha x)$$

$$\Leftrightarrow l\left(\frac{\alpha}{\alpha + \beta} y + \frac{\beta}{\alpha + \beta} \tilde{y}\right) \leq \frac{\alpha}{\alpha + \beta} p(y - \beta x) + \frac{\beta}{\alpha + \beta} p(\tilde{y} + \alpha x).$$

But: hypothesis on l .

$$l\left(\frac{\alpha}{\alpha + \beta} y + \frac{\beta}{\alpha + \beta} \tilde{y}\right) \leq p\left(\frac{\alpha}{\alpha + \beta} (y - \beta x) + \frac{\beta}{\alpha + \beta} (\tilde{y} + \alpha x)\right) \leq$$

convexity of p

$$\leq \frac{\alpha}{\alpha + \beta} p(y - \beta x) + \frac{\beta}{\alpha + \beta} p(\tilde{y} + \alpha x)$$

Thus: $I_{y; \alpha, \beta} \neq \emptyset$

$$\xrightarrow{\text{Any}} \text{Finite intersection} \quad \bigcap_{k=1}^N I_{y; \alpha_k, \beta_k} \neq \emptyset.$$

By finite intersection property of compact sets $\Rightarrow \bigcap_{y \in Y} \bigcap_{\alpha, \beta > 0} I_{y; \alpha, \beta} \neq \emptyset$

To conclude:

We showed:

$$\exists \lambda \in \bigcap_{y \in \bar{Y}} \bigcap_{\alpha, \beta > 0} \left[\frac{l(y) - p(y) - \beta x}{\beta}, \frac{p(y + \alpha x) - l(y)}{\alpha} \right]$$

→ For such λ :

$$\tilde{l}(y + \alpha \cdot x) \leq p(y + \alpha \cdot x), \forall y \in Y, \alpha \in \mathbb{R}$$

where $\tilde{l}(y + \alpha \cdot x) = l(y) + \alpha \cdot \lambda$

End of **LEMMA**

Step 2: **MAXIMALITY** By Zorn's Lemma.

Consider pairs $\mathcal{P} = \{(\tilde{Y}, \tilde{L})\}$ of linear subspaces $\tilde{Y} \subset \bar{X}$ and linear maps $\tilde{L}: \tilde{Y} \rightarrow \mathbb{R}$ such that:

- (1) $Y \subset \tilde{Y}$
- (2) $\tilde{L}(z) \leq p(z), \forall z \in \tilde{Y}, (3) \tilde{L}|_Y = l$

and: (4) If $(\tilde{Y}_1, \tilde{L}_1), (\tilde{Y}_2, \tilde{L}_2) \in \mathcal{P}$:

$$\tilde{L}_1|_{\tilde{Y}} = \tilde{L}_2|_{\tilde{Y}} \text{ where } \tilde{Y} = \tilde{Y}_1 \cap \tilde{Y}_2 \}$$

On \mathcal{P} there is a partial order:

$$(\tilde{Y}_1, \tilde{L}_1) \triangleleft (\tilde{Y}_2, \tilde{L}_2) \text{ if } \tilde{Y}_1 \subset \tilde{Y}_2.$$

→ Extract an ascending chain: $\{(\tilde{Y}_\alpha, \tilde{L}_\alpha), \alpha \in A\}$.

such that $\forall \alpha, \beta \in A$ either $(\tilde{Y}_\alpha, \tilde{L}_\alpha) \triangleleft (\tilde{Y}_\beta, \tilde{L}_\beta)$ or $(\tilde{Y}_\beta, \tilde{L}_\beta) \triangleleft (\tilde{Y}_\alpha, \tilde{L}_\alpha)$

Such ascending chains are not empty, because (\bar{Y}, ℓ) is in each. (7)

By Zorn's lemma: \mathcal{P} has a maximal element (or a maximal upper bound): $\exists (Y_0, L_0) \in \mathcal{P}$ such that.

If $(Y_1, L_1) \in \mathcal{P}$ with $(Y_0, L_0) \triangleleft (Y_1, L_1)$ then $(Y_1, L_1) = (Y_0, L_0)$

Let \mathcal{C} be an ascending chain: $\mathcal{C} \subset \mathcal{P}$

Then define: $\tilde{Y} = \bigcup \tilde{Y}_\alpha$ Note: \tilde{Y} is a linear space
where: $(\tilde{Y}_\alpha, \tilde{L}_\alpha) \in \mathcal{C}$ $\tilde{Y} \subset \bar{X}$

and: $\tilde{L}: \tilde{Y} \rightarrow \mathbb{R}$: if $x \in \tilde{Y}$, then $\exists \alpha$ s.t. $x \in \tilde{Y}_\alpha$

$\perp \rightarrow \tilde{L}(x) = \tilde{L}_\alpha(x)$

linear map.

Claim: (\tilde{Y}, \tilde{L}) is a maximal element:

Why: $\forall (\tilde{Y}_\alpha, \tilde{L}_\alpha) \in \mathcal{C}$: $\tilde{Y}_\alpha \subset \tilde{Y}$ and $\tilde{L}|_{\tilde{Y}_\alpha} = \tilde{L}_\alpha$

By Zorn's lemma: There exists a maximal upper bound: $(Y_0, L_0) \in \mathcal{P}$

such $\forall (Y_1, L_1) \in \mathcal{P}$ with $(Y_0, L_0) \triangleleft (Y_1, L_1)$ then $Y_1 = Y_0, L_1 = L_0$

Therefore: $Y_0 = \bar{X}$: $\rightarrow L_0$ satisfies the Hahn-Banach theorem □

(8)

Use Case : $(\underline{X}, \|\cdot\|)$ is a normed vector space.

Let $\ell: Y \subset \underline{X} \rightarrow \mathbb{R}$ a bounded (continuous) linear functional on $Y: \|\ell(y)\| \leq C \cdot \|y\|$

$\rightarrow \exists L: \underline{X} \rightarrow \mathbb{R}: |L(x)| \leq C \cdot \|x\|, \forall x \in \underline{X}$
 $L|_Y = \ell.$

$\rightarrow \|L\| = \|\ell\|$ & $L|_Y = \ell.$

$\left[\dots \rightarrow \underline{X} \hookrightarrow \underline{X}^{**} \text{ isometric embedding. } \right]$