(L8)

Topologies. Dual Topologies.

Let $\underline{X}$ be a set. We denote by $\mathcal{P}(\underline{X})$ the collection of all subsets of $\underline{X}$.

Definition A collection τ of subsets of $\underline{X}$ is called a topology on $\underline{X}$ if it satisfies:

(1) $\phi \in \tau$.

(2) $\underline{X} \in \tau$.

(3). If. $\{A_\alpha\}_{\alpha \in A}$ is an arbitrary collection of elements of τ, i.e., $A_\alpha \in \tau, \forall \alpha$, then $\bigcup_{\alpha \in A} A_\alpha \in \tau$.

(4). If. $A_1 \dots A_N \in \tau$ then $A_1 \cap \dots \cap A_N \in \tau$.

Elements of τ are called open sets. The pair $(\underline{X}, \tau)$ is called a topological space.

Example. Assume $(\underline{X}, d)$ is a metric space.

$$\tau = \tau_d = \{A \mid A \subset \underline{X} : \forall x \in A \ \exists r > 0 \text{ s.t. } B_r(x) \subset A\}.$$

↳ metric-induced topology.

Definitions: Assume $(\underline{X}, \tau)$ is a topological space.

a) $A \subset \underline{X}$ is said closed if $A^c := \underline{X} \setminus A \in \tau$

b) $A \subset \underline{X}$ is said compact if

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$[\ \forall \{E_\alpha : \alpha \in A\} \subset \tau$ s.t. $A \subset \bigcup_{\alpha \in A} E_\alpha$, $\exists J \subset A$, $|J| < \infty$ s.t. $A \subset \bigcup_{\alpha \in J} E_\alpha$.

c) $(\underline{X}, \tau)$ is said Hausdorff (or T_2) if:

$\forall\, x,y \in \underline{X},\ x \neq y,\ \exists\, A, B \in \tau : A \cap B = \phi,\ x \in A, y \in B.$

d). A set A is called a neighborhood of $x \in \underline{X}$ if there exits $E \in \tau$ s.t. $x \in E \subset A$.

Switch back to $(\underline{X}, \|\cdot\|)$ a normed linear space. (NLS).

Topologies:

① Norm-induced topology:

$$\|\cdot\| \longrightarrow d(x,y) = \|x-y\| \longrightarrow \tau_d = \tau_{\|\cdot\|}.$$

Definition: Assume $(\underline{X}, \tau)$ is a topological space.

(a) A collection. $\mathcal{B} \subset \tau$ is called base for topology τ if. $\forall x \in \underline{X},\ \forall A \in \tau$, s.t. $x \in A$, $\exists\, U \in \mathcal{B}$ s.t. $x \in U \subset A$

(b) A collection. $\mathcal{S} \subset \tau$ is called a subbase for topology τ if $\{A_1 \cap A_2 \cap \ldots \cap A_N,\ \forall N \geq 1 \text{ integer},\ \forall A_1 \ldots A_N \in \mathcal{S}\}$ is a base for τ.

Norm-induced topology: Base $\mathcal{B} = \{B_q(x) : x \in \underline{X},\ q \in \mathbb{Q}^+\}$.

$$B_r(x) = \{y \in \underline{X} : \|x-y\| < r\}$$

Convergence:

$x_n \longrightarrow x$, in norm-sense:

$$\lim_{n\to\infty} \|x_n - x\| = 0.$$

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② <u>Weak topology</u>: Denoted. $\sigma(\underline{X}, \underline{X}^*)$.

dual space $= \{ \ell : \underline{X} \to \mathbb{C},$ ℓ continuous $\}$ (ℓ bounded).

Subbase for topology:

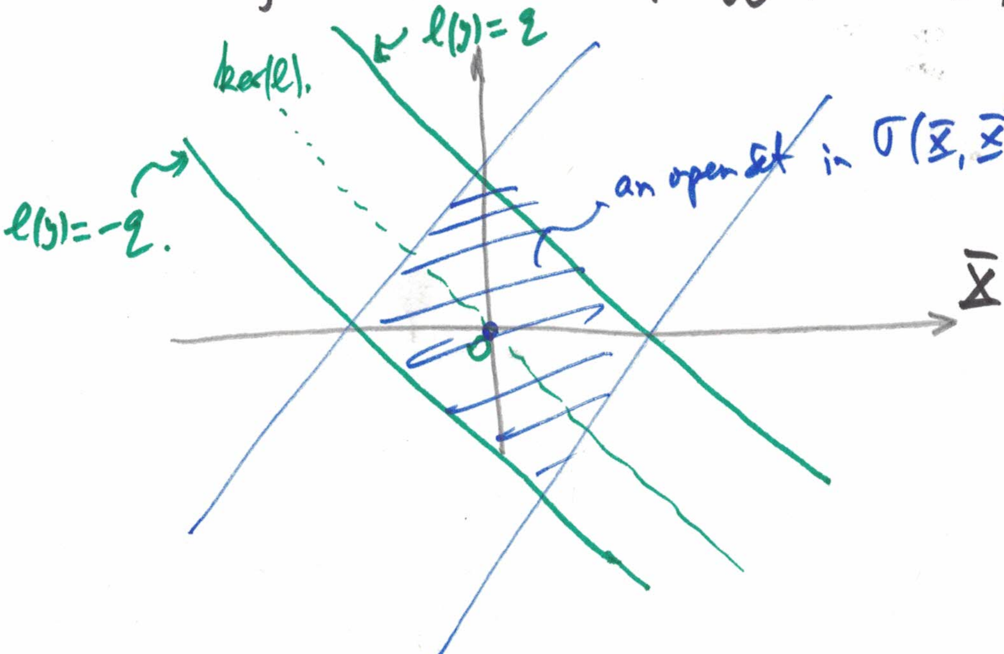
$\forall \ell \in \underline{X}^*$, $\forall \varepsilon \in \mathbb{Q}^+$, $\forall z \in \underline{X}$

$$V_z(\ell, \varepsilon) = \{ x \in \underline{X} : |\ell(x) - \ell(z)| < \varepsilon \} =$$

$$= z + \{ y \in \underline{X} : |\ell(y)| < \varepsilon \} = z + \underbrace{V(\ell, \varepsilon)}_{\text{subset of } \underline{X}}.$$

Collection $\{ V_z(\ell, \varepsilon) : z \in \underline{X}, \ell \in \underline{X}^*, \varepsilon \in \mathbb{Q}^+ \}$

is a subbase for the weak topology, $\sigma(\underline{X}, \underline{X}^*)$.



Convergence in weak topology sense:

$$x_n \xrightarrow{w} x \quad , \quad \text{weak convergence:}$$

$$\forall \ell \in \underline{X}^*, \quad \lim_{n\to\infty} |\ell(x_n - x)| = 0. \quad \Big\{ \Leftrightarrow$$

$$\lim_{n\to\infty} \ell(x_n) = \ell(x).$$

$(\underline{X}, \|\cdot\|) \to \tau_{\|\cdot\|}$ topology $\longrightarrow \underline{X}^* = \{ \ell : \underline{X} \to \mathbb{C}$, ℓ cont. w.r.t. $\tau_{\|\cdot\|}$

$\longrightarrow \sigma(\underline{X}, \underline{X}^*)$ is the smallest topology s.t. $\forall \ell \in \underline{X}^*$ remains continuous.

Prop. If. $x_n \to x$ in norm then $x_n \xrightarrow{w} x$

Proof: $\forall \ell \in \underline{X}^*$

$$0 \le |\ell(x_n - x)| \le \|\ell\| \cdot \underbrace{\|x_n - x\|}_{\to 0,\ n\to\infty}. \quad \Rightarrow \quad x_n \xrightarrow{w} x.$$

⊠

③ Weak $*$ topology: Denoted $\sigma(\underline{X}^*, \underline{X})$.

Subbase for topology: $\{ V_{\ell_0}(x, \varepsilon) : \forall \ell_0 \in \underline{X}^*, \forall x \in \underline{X}, \forall \varepsilon \in \mathbb{Q}^+ \}$.

$\forall x \in \underline{X}, \ \forall \varepsilon \in \mathbb{Q}^+, \ \forall \ell_0 \in \underline{X}^* \quad \Big\}$

$$V_{\ell_0}(x, \varepsilon) := \{ \ell \in \underline{X}^* : \ |\ell(x) - \ell_0(x)| < \varepsilon \} =$$

$$= \ell_0 + V_0(x, \varepsilon). = \ell_0 + \{ \ell \in \underline{X}^* : |\ell(x)| < \varepsilon \}.$$

<u>Proposition</u> If. $(\underline{X}, \|\cdot\|)$ is <u>reflexive</u> a ~~Ba~~ Banach space: $\underline{X}^{**} \simeq \underline{X}$

then. the weak topology on $\underline{X}$ coincides with the weak $*$ topology on $\underline{X}$ (pull-back from $\underline{X}^{**}$).

Convergence in <u>weak $*$ - sense</u> :

$l_n \to l.$ in weak $*$ sense if:

$$\forall x \in \underline{X} : \quad \lim_{n\to\infty} l_n(x) = l(x).$$

The <u>dual operator</u> :

let $(\underline{X}, \|\cdot\|)$, $(Y, |||\cdot|||)$ be two normed linear spaces

and. $A: \underline{X} \to \underline{Y}$ a bounded linear operator :

$$\|A\|_{B(\underline{X},Y)} = \sup_{\|x\|=1} |||Ax||| < \infty$$

<u>Definition</u> : ${}^tA : \underline{Y}^* \to \underline{X}^*$ be the linear operator,

$${}^tA(l).(x) = l(A(x)), \quad \forall l \in \underline{Y}^*, \ \forall x \in \underline{X}.$$

${}^tA$ is called <u>the dual of A</u> (the "transpose" of A).

$$\begin{array}{ccc} \underline{X} & \xrightarrow{A} & \underline{Y} \\ \downarrow & & \downarrow \\ \mathbb{C} & & \mathbb{C} \end{array} \quad \dashrightarrow \quad \underline{X}^* \xleftarrow{\ {}^tA\ } \underline{Y}^*$$

Proposition

$$\| {}^tA \|_{B(Y^*, X^*)} = \| A \|_{B(X,Y)}.$$

Proof.

$$\| {}^tA \| = \sup_{\substack{l \in Y^* \\ \|l\|_{Y^*} = 1}} \| {}^tA(l) \|_{X^*} = \sup_{\substack{l \in Y^* \\ \|l\|_{Y^*} = 1}} \sup_{\substack{x \in X \\ \|x\| = 1}} | {}^tA(l)(x) | =$$

$$= \sup_{\substack{l \in Y^* \\ \|l\|_{Y^*} = 1}} \sup_{\substack{x \in X \\ \|x\| = 1}} | l(A(x)) |. = \sup_{\substack{x \in X \\ \|x\| = 1}} \sup_{\substack{l \in Y^* \\ \|l\|_{Y^*} = 1}} | l(Ax) |. \overset{\text{by H-B.}}{=}$$

$$= \sup_{\substack{\|x\| = 1 \\ x \in X}} \| Ax \|_Y = \| A \|_{B(X,Y)}. \qquad \square$$

${}^tA$ is similar to A^*, when X, Y are Hilbert spaces, except for complex conjugation.

$$A^*: \quad \langle Ax, y \rangle = \langle x, A^* y \rangle.$$

Let $C: H \to H$, the antilinear map s.t. $\langle x, y \rangle = \langle C(y), C(x) \rangle$.

Then: $\quad C(A^* x) = {}^tA \cdot C(x).$

let $(\mathbb{X}, \|\cdot\|)$, $(Y, |||\cdot|||)$ be two normed linear spaces. (7).

$$B(\mathbb{X}, Y) = \{ A : X \to Y,\ A \text{ linear} : \|A\| = \sup_{\|x\|=1} |||Ax||| < \infty$$

Topologies on $B(\mathbb{X}, Y)$:

① Uniform topology / Uniform convergence:

$\downarrow$

$\|\cdot\|_{B(X,Y)}$ -induced topology:

$A_n \to A$ uniform topology:

$$\lim_{n\to\infty} \|A_n - A\| = 0 \quad : \quad \lim_{n\to\infty} \sup_{\|x\|=1} |||(A_n - A)x||| = 0.$$

② Strong topology / Strong convergence:

$A_n \xrightarrow{s} A$: strong convergence:

$$\forall x \in \mathbb{X}, \quad \lim_{n\to\infty} |||(A_n - A)x||| = 0.$$

Weakest topology on $B(\mathbb{X}, Y)$ s.t.

$$\forall x \in \mathbb{X} : \quad A \in B(\mathbb{X}, Y) \xmapsto{E_x} Ax \in Y.$$

are continuous.

③ Weak topology / weak convergence:

$A_n \xrightarrow{w} A$: weak convergence:

$$\forall x \in \mathbb{X},\ \forall \ell \in Y^* : \quad \lim_{n\to\infty} |\ell(A_n(x)) - \ell(A(x))| = 0.$$

Weak topology on $B(X,Y)$ is the weakest topology such that:

$$\forall x \in X, \quad \forall \ell \in Y^* : \quad A \in B(X,Y) \xmapsto{E_{x,\ell}} \ell(A(x)) \in \mathbb{C}$$

$E_{x,\ell}$ are continuous.