# The Baire Category Theorem & Consequences



**Setup:** $(X, d)$ is a metric space.

$\tau = \tau_d$ is the metric induced topology.

**Definition** A set $A \subset X$ is said **nowhere dense** if its closure has empty interior: $\operatorname{int}(\overline{A}) = \phi$.

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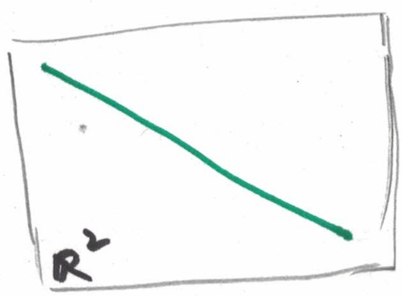
**Examples:**

1. $\mathbb{N} \subset \mathbb{R}$, $\mathbb{N}$ is nowhere dense.
2. $\mathbb{Q} \subset \mathbb{R}$, $\mathbb{Q}$ is NOT nowhere dense: $\overline{\mathbb{Q}} = \mathbb{R}$.

   ↑ with $|\cdot|$ distance.

3. Line $\subset \mathbb{R}^2$




$\operatorname{int}(\text{Line}) = \phi$.

$\text{line} = \overline{\text{line}}$.

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**Definition.** A topological space $(X, \tau)$ is said to be **of $1^{st}$ (Baire) category** if: there are countably many $A_n \subset X$ nowhere dense sets s.t. $X = \bigcup_{n=1}^{\infty} A_n$.

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**Definition** A topological space $(X, \tau)$ is said to be of (Baire) $2^{nd}$ category if it is not of $1^{st}$ category.

[In other words, a second category topological space cannot be written as a countable union of nowhere dense sets.

**Theorem** [Baire Category Theorem] Any complete metric space is of (Baire) $2^{nd}$ category.

**Proof.**

By contradiction. Assume $(X, d)$ is a complete metric space and $(A_n)_{n\geq 1}$, $A_n \subset X$ are nowhere dense sets, i.e., $\operatorname{int}(\overline{A_n}) = \phi$ and $X = \bigcup_{n\geq 1} A_n$.

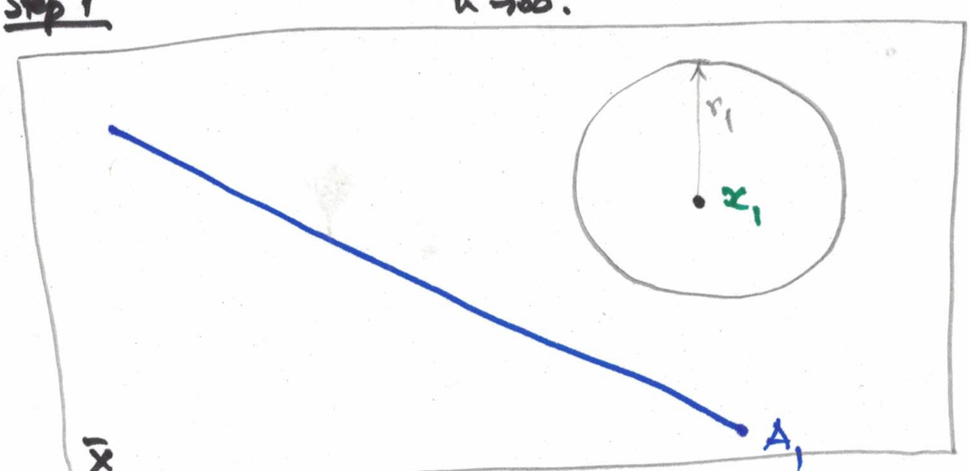
Idea: Construct $(x_n)_{n\geq 1}$ a Cauchy sequence s.t. $x_n \notin \bigcup_{k=1}^{n} \overline{A_k}$

$$d\left(x_n, \bigcup_{k=1}^{n} \overline{A_k}\right) > 0.$$

$\dashrightarrow$ 1) $\lim_{n\to\infty} x_n \notin \bigcup_n A_n$ $\rightarrow$ contradiction.

2) $\lim_{n\to\infty} x_n \in X$.

Step 1




$X \setminus \overline{A_1}$ open. non-empty set

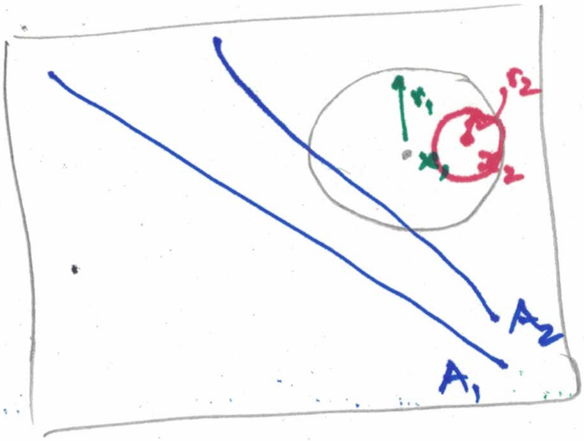
$\exists\, r_1 > 0$

$B_{r_1}(x_1) \subset X \setminus \overline{A_1}$

choose $r_1 \leq 1$.

$B_{r_1}(x_1) \cap A_1 = \phi$.

<u>Step 2</u>




$X \setminus \overline{A_2}$ non-empty open set.

$(X \setminus \overline{A_2}) \cap B_{r_1}(x_1) \neq \phi$.

<u>why</u>:

Alternatively: take the complement.

$\overline{A_2} \cup (X \setminus B_{r_1}(x_1)) = X$.

$\to B_{r_1}(x_1) \subset \overline{A_2} \to$

$\to$ is interior of $\overline{A_2}$ is not empty $\to$ contr

Take $x_2 \in (X \setminus \overline{A_2}) \cap B_{r_1}(x_1)$

$= B_{r_1}(x_1) \setminus \overline{A_2}$

s.t. $\exists\, r_2 > 0$,

1. $\overline{B_{r_2}(x_2)} \subset B_{r_1}(x_1)$.
2. $B_{r_2}(x_2) \subset X \setminus \overline{A_2}$
3. $r_2 \le \frac{r_1}{2} \le \frac{1}{2}$.

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Proceed inductively:

At <u>step n</u>: Take $x_n \in (X \setminus \overline{A_n}) \cap B_{r_{n-1}}(x_{n-1}) =$

$= B_{r_{n-1}}(x_{n-1}) \setminus \overline{A_n}$

and choose $r_n > 0$ s.t.

1) $\overline{B_{r_n}}(x_n) \subset B_{r_{n-1}}(x_{n-1})$

2) $B_{r_n}(x_n) \subset X \setminus \overline{A_n}$

3) $r_n \le \frac{r_{n-1}}{2} \le \frac{1}{2^{n-1}}$.

Claim 1: $(x_n)_{n\geq 1}$ is Cauchy.

Why: Take. $n, m \geq N$:

$$d(x_n, x_m) \leq d(x_n, x_N) + d(x_m, x_N) \leq r_N + r_N =$$

$$= 2 r_N \leq \frac{2}{2^{N-1}} = \frac{1}{2^{N-2}}$$

$$\Rightarrow \lim_{N\to\infty} \sup_{n,m\geq N} d(x_n, x_m) = 0. \rightarrow (x_n)_{n\geq 1} \text{ is Cauchy.}$$

$(X, d)$ complete $\Rightarrow z = \lim_{n\to\infty} x_n \in X$.

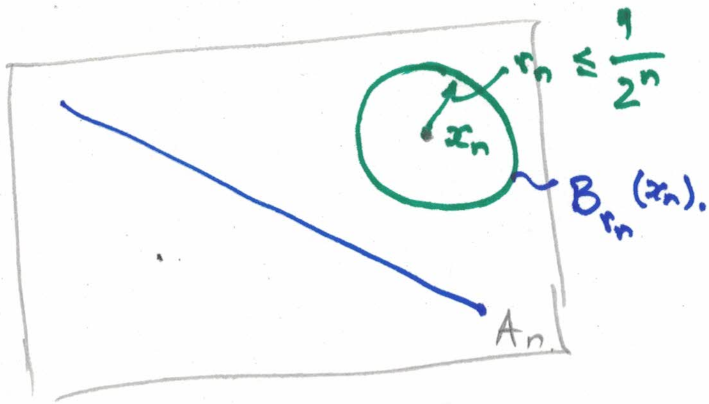
Claim 2. $z \notin A_n$, for every $n \geq 1$.

Why: ~~Fix N.~~

Fix n.

$(x_m)_{m\geq n}$:

$x_m \in B_{r_{n+1}}(x_{n+1})$, $\forall m \geq n+1$




$$\Rightarrow z = \lim_{m\to\infty} x_m \in \overline{B_{r_{n+1}}(x_{n+1})} \subset B_{r_n}(x_n).$$

We know: $B_{r_n}(x_n) \subset X \setminus \overline{A_n} \Rightarrow B_{r_n}(x_n) \cap A_n = \phi$.

$$\longrightarrow z \notin A_n.$$

If $\overline{X = \bigcup_{n\geq 1} A_n}$ ~~and~~ $z \in X \Rightarrow \exists N$ s.t. $z \in A_N$

Contradiction with claim 2.

Main Application:

Theorem [Banach – Steinhaus Theorem, Or the Principle of Uniform Boundedness]. Let $(\underline{X}, \|\cdot\|)$ be a Banach space. Let $\mathcal{F}$ be a family of bounded linear maps from $\underline{X}$ to some normed linear space $Y$. Suppose for each $x \in \underline{X}$ the set $\{\|Tx\|, T \in \mathcal{F}\}$ is bounded. Then $\{\|T\|, T \in \mathcal{F}\}$ is bounded.

In other words:
1) $\underline{X}$ is complete.
2) $\mathcal{F} \subset B(\underline{X}, Y)$.
~~3) $\forall x \in \underline{X}, \{\|Tx\|, T \in \mathcal{F}$~~
3) $\forall x \in \underline{X}\ \exists M_x > 0$ s.t. $\forall T \in \mathcal{F}, \|Tx\| \le M_x$

$\Rightarrow \exists C > 0$ s.t. $\forall x \in \underline{X}, \forall T \in \mathcal{F}, \|Tx\| \le C \cdot \|x\|$.

**Proposition.** Let $\underline{X}$ and $\underline{Y}$ be two normed linear spaces.

Let $T : \underline{X} \to \underline{Y}$ be a linear map.

① If $T$ is bounded, then

$$T^{-1}\left(\overline{B_1(0)}\right) = \{x \in \underline{X} : \|Tx\|_Y \leq 1\}.$$

has non-empty interior.

② If $T^{-1}\left(\overline{B_1(0)}\right)$ has non-empty interior, then $T$ is bounded. Furthermore, if $B_{r_0}(x_0) \subset T^{-1}\left(\overline{B_1(0)}\right)$ then:

$$\|T\|_{B(\underline{X},Y)} \leq \frac{1+\|T(x_0)\|_Y}{r_0}$$

Pf:

①: If $T$ is bounded $\to$ $T$ is continuous $\to$

$T^{-1}(B_1(0))$ is open.
$\{0\} \in T^{-1}(B_1(0))$ $\to$ $T^{-1}(B_1(0))$ is a not empty open set.

$\to T^{-1}\left(\overline{B_1(0)}\right)$ has non-empty interior.

② Assume $B_{r_0}(x_0) \subset T^{-1}\left(\overline{B_1(0)}\right)$ :

$\forall x \in \underline{X}$, if $\|x - x_0\|_{\underline{X}} < r_0 \Rightarrow \|Tx\|_Y \leq 1$.

Take. $y \in \underline{X}$, $\|y\| < r_0$.

$$Ty = T(\underbrace{y+x_0}_{\in B_{r_0}(x_0)}) - T(x_0).$$

$$\|Ty\| \le \|T(y+x_0)\| + \|T(x_0)\| \le 1 + \|T(x_0)\|.$$

$$\Rightarrow \cancel{\left\| T\frac{y}{\|y\|}\right\| \le}$$

$$\sup_{\|y\| \le r_0} \|Ty\| \le 1 + \|T(x_0)\|.$$

If. $x \in \underline{X}$, $\|x\| < 1$: $\quad x = \cancel{\frac{r_0}{r_0}} \frac{1}{r_0} \cdot \underbrace{(r_0 x)}_{\|r_0 x\| < r_0}$.

$$\Rightarrow \|Tx\| = \frac{1}{r_0} \|T(r_0 x)\| \le \frac{1 + \|T(x_0)\|}{r_0}.$$

$$\Rightarrow \sup_{\|x\| \le 1} \|Tx\| \le \underbrace{\frac{1 + \|T(x_0)\|}{r_0}} < \infty. \Rightarrow T \text{ bounded}.$$

$$\|T\|_{B(X,Y)} \le \frac{1 + \|T(x_0)\|}{r_0}:$$