Final Exam

December 14, 2007 , 8am-11am, MATH 1311

You are allowed one two-sided cheat sheet with formulae/text of your choice. No discussions are allowed.

Problem 1. (20pts)
Consider the following system of differential equations in plane $\boldsymbol{R}^2$,

$$\dot{x} = x - y - x(x^2 + y^2) + \frac{xy}{\sqrt{x^2 + y^2}}$$

$$\dot{y} = x + y - y(x^2 + y^2) - \frac{x^2}{\sqrt{x^2 + y^2}}$$

a. (10pt) Draw its phase portrait.
b. (10pt) Based on the phase portrait obtained before, analyze the stability behavior at its two fixed points.

Problem 2. (25pts)
Consider $U(x) = \frac{1}{2} x^T Hx$ with H a dxd real symmetric matrix and its associated second order differential equation,

$$\ddot{x} = -\nabla U(x)$$

a. (5pt) Rewrite this equation as a system of linear first order differential equations

$$\dot{z} = Az$$

b. (5pt) Show that $J^{-1}AJ = -A^T$ where

$$J = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$$

c. (5pt) Prove that, if λ is an eigenvalue of A, then so is $-\lambda$.
d. (5pt) Can the origin 0 be positively asymptotically stable?
e. (5pt) Can the origin be positively stable?

Note: For points d & e you need to argue your answer: For a positive answer, you need to provide an example; for a negative answer, you need to prove the claim.

Problem 3. (10pt)
Solve the following initial value problem

$$\dot{x} = x \ln x + xe^t$$
$$x(0) = 1$$

Problem 4. (10pt)
Solve the following initial value problem

$$\ddot{x} - 3\dot{x} + 2x = e^t$$
$$x(0) = 0, \dot{x}(0) = 0$$

Total: 65 points