Chapter 6.6 "Least-squares Line"

Given an experimental data

$$(x_1,y_1),(x_2,y_2),\dots,(x_n,y_n)$$

we want to find a line $y=\beta_0+\beta_1 x$ that best fits the data. In particular, we want β_0 and β_1 such that

$$\begin{aligned}\beta_0+\beta_1 x_1 &= y_1\\ \vdots \quad & \quad \vdots\\ \beta_0+\beta_1 x_n &= y_n\end{aligned}$$

This is same as trying to solve the linear system

$$\underbrace{\begin{bmatrix}1 & x_1\\ 1 & x_2\\ \vdots & \vdots\\ 1 & x_n\end{bmatrix}}_{=A}\underbrace{\begin{bmatrix}\beta_0\\ \beta_1\end{bmatrix}}_{=\beta}=\underbrace{\begin{bmatrix}y_1\\ y_2\\ \vdots\\ y_n\end{bmatrix}}_{=b}$$

In many real life applications, $A\beta=b$ is inconsistent. The least-squares solution $\begin{bmatrix}\beta_0 & \beta_1\end{bmatrix}^T$ defines a line $y=\beta_0+\beta_1 x$ which we call **least-squares line** that best fits the data point $(x_1,y_1),\dots,(x_n,y_n)$. To recall, $\begin{bmatrix}\beta_0 & \beta_1\end{bmatrix}^T$ is the solution to the equation $A^TA\beta=A^Tb$.

Chapter 7.1 Diagonalization of Symmetric Matrices

Defintion 1. A matrix A is **symmetric** if $A=A^T$. Equivalently, the matrix has arbitrary entries along the main diagonal, and its entries are symmetric with respect to the main diagonal.

Defintion 2. A is **orthogonally diagonalizable** if there exists an orthogonal matrix P ($P^{-1}=P^D$) and a diagonal matrix D such that $A=PDP^T$.

Theorem 3 (Spectral Theorem)**.** Let A be an symmetric $n\times n$-matrix. Then

(a) A has n real eigenvalues, counting multiplicities.

(b) The dimension of the eigenspace for each eigenvalue λ equals the multiplicity of λ as a root of the characteristic equation. (i.e. diagonalizable)

(c) The eigenspaces are mutually orthogonal, in the sense that eigenvectors corresponding to different eigenvalues are orthogonal.

Furthermore, an $n\times n$-matrix A is symmetric if and only if A is orthogonally diagonalizable.

How do we orthogonally diagonalize an $n\times n$-matrix A? You can do this when you can find an orthonormal basis consisting $\{u_1,\dots,u_n\}$ of eigenvectors of A (not always possible). Let

$$Q=\begin{bmatrix}u_1 & \cdots & u_n\end{bmatrix}$$

which is an orthogonal matrix. Then

$$A=QDQ^T$$

where D is the diagonal matrix with eigenvalues corresponding to $\{u_1,\dots,u_n\}$. What is **amazing** about the spectral theorem is that it says that for a symmetric matrix A, you can always find an orthonormal basis of eigenvectors of A.