Spring 2019 - Solution

1. (a) Consider the linear system $\begin{cases} x_1 + \ x_2 = 3 \\ 4x_1 + 2x_2 = 4. \\ 3x_1 - 3x_2 = 5 \end{cases}$

i. (8 points) Show that this linear system is inconsistent.

ii. (8 points) Find the least-squares solution(s) to this linear system.

i. We row reduce the augmented matrix

$$\begin{bmatrix} 1 & 1 & 3 \\ 4 & 2 & 4 \\ 3 & -3 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -8 \\ 0 & -6 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 3 \\ 0 & -2 & -8 \\ 0 & 0 & 20 \end{bmatrix}$$

The last column is a pivot column, so the system is inconsistent.

1. (a) Consider the linear system $\begin{cases} x_1 + \ x_2 = 3 \\ 4x_1 + 2x_2 = 4. \\ 3x_1 - 3x_2 = 5 \end{cases}$

i. (8 points) Show that this linear system is inconsistent.

ii. (8 points) Find the least-squares solution(s) to this linear system.

ii.

$$A = \begin{bmatrix} 1 & 1 \\ 4 & 2 \\ 3 & -3 \end{bmatrix} \text{ and } b = \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} \quad \Rightarrow \quad \text{We solve } A^T Ax = A^T b$$

$$A^T A = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & 2 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} 26 & 0 \\ 0 & 14 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 4 & 3 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 34 \\ -4 \end{bmatrix}$$

1. (a) Consider the linear system $\begin{cases} x_1 + \ \ x_2 = 3 \\ 4x_1 + 2x_2 = 4 \\ 3x_1 - 3x_2 = 5 \end{cases}$.

i. (8 points) Show that this linear system is inconsistent.

ii. (8 points) Find the least-squares solution(s) to this linear system.

ii. The corresponding augmented matrix is

$$\begin{bmatrix} 26 & 0 & 34 \\ 0 & 14 & -4 \end{bmatrix}$$

$$\begin{aligned} x_1 &= \frac{34}{26} = \frac{17}{13} \\ x_2 &= -\frac{4}{14} = -\frac{2}{7} \end{aligned}$$

(b) Answer TRUE or FALSE. You do not have to justify your answers.

i. (4 points) Let H denote the set of all polynomials in $\mathbb{P}_2$ whose coefficients are all greater than or equal to 0. Then H is a subspace of $\mathbb{P}_2$.

Consider $t \in H$. However, $-1 \cdot t$ is not in H. Therefore, H is not closed under scalar multiplication. **False**

ii. (4 points) Let K denote the set of all vectors in $\mathbb{R}^3$ with the property that the sum of their three entries equals 0. Then K is a subspace of $\mathbb{R}^3$.

We can rewrite

$$K = \left\{ \begin{bmatrix} x & y & z \end{bmatrix}^T \in \mathbb{R}^3 \mid x + y + z = 0 \right\} = \mathrm{Null}\left(\begin{bmatrix} 1 & 1 & 1 \end{bmatrix} \right)$$

The null space of a 1×3 matrix is always a subspace of $\mathbb{R}^3$. **True**

iii. (4 points) The set $\mathbb{P}$ of **all** polynomials in the variable t is an infinite-dimensional vector space.

Suppose $\mathbb{P}$ is a infinite-dimensional vector space with a basis

$$\{1, t, t^2, t^3, \ldots\}$$

which contains infinitely many elements. **True**

2. (a) (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first rotates vectors in $\mathbb{R}^2$ by $45°$ counterclockwise, then reflects across the x_1-axis. Compute the standard matrix for T, and then compute the result when T is applied to the vector $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

The matrix corresponding to rotating vectors in $\mathbb{R}^2$ by 45 counterclockwise is

$$\begin{bmatrix} \cos(45) & -\sin(45) \\ \sin(45) & \cos(45) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

The matrix corresponding to reflecting across the x_1-axis is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

The standard matrix for T is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}$$

2. (a) (10 points) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation that first rotates vectors in $\mathbb{R}^2$ by $45°$ counterclockwise, then reflects across the x_1-axis. Compute the standard matrix for T, and then compute the result when T is applied to the vector $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$.

The result when T is applied to the vector $\begin{bmatrix} -2 \\ 4 \end{bmatrix}$

$$\begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix} = \begin{bmatrix} -\sqrt{2} - 2\sqrt{2} \\ \sqrt{2} - 2\sqrt{2} \end{bmatrix} = \begin{bmatrix} -3\sqrt{2} \\ -\sqrt{2} \end{bmatrix}$$

(b) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix}$.

i. (10 points) Compute $\det A$ and $\det(A^3)$.

ii. (8 points) Compute A^{-1} or explain why it doesn't exist.

i.

$$\begin{aligned} \det A &= \det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix} = 1 \cdot \begin{vmatrix} 2 & 0 \\ 0 & 4 \end{vmatrix} + \cdot 1 \cdot \begin{vmatrix} 0 & 2 \\ 3 & 0 \end{vmatrix} \\ &= 8 - 6 = 2 \end{aligned}$$

$$\det(A^3) = \det(A)^3 = 2^3 = 8$$

(b) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 3 & 0 & 4 \end{bmatrix}$.

i. **(10 points)** Compute $\det A$ and $\det(A^3)$.

ii. **(8 points)** Compute A^{-1} or explain why it doesn't exist.

ii. Since $\det(A) = 2 \neq 0$, A is invertible.

$$\left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 3 & 0 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right]$$

$$\to \left[\begin{array}{cccccc} 1 & 0 & 0 & 4 & 0 & -1 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 & -3 & 0 & 1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 4 & 0 & -1 \\ 0 & \frac{1}{2} & 0 \\ -3 & 0 & 1 \end{bmatrix}$$

3. Consider the vector space $V = \mathbb{P}_2$ and the basis $\mathcal{B} = \{1,\ t,\ t^2\}$ for $\mathbb{P}_2$.

(a) (3 points) TRUE or FALSE: $\dim(\mathbb{P}_2) = 3$.

True since $\mathbb{P}_2$ has a basis $\{1, t, t^2\}$.

(b) (3 points) TRUE or FALSE: $\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$.

Since $\dim(\mathbb{P}_2) = 3$, $\mathbb{P}_2$ is isomorphic to $\mathbb{R}^3$. **True** (For more detail, see Theorem 9 on pg 235 and the first paragraph in pg 236)

(c) (4 points) Find the $\mathcal{B}$-coordinate vector of $3t - t^2$.

As $3t - t^2 = 0 \cdot 1 + 3 \cdot t + (-1) \cdot t^2$,

$$[3t - t^2]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}$$

3. Consider the vector space $V = \mathbb{P}_2$ and the basis $\mathcal{B} = \{1,\ t,\ t^2\}$ for $\mathbb{P}_2$.

(d) (8 points) Let $T : \mathbb{P}_2 \to \mathbb{P}_2$ be the linear transformation defined by the formula $T(p(t)) = p(t) - p'(t)$. [Here, $p'(t)$ denotes the derivative of $p(t)$.] Find the matrix $[T]_{\mathcal{B}}$ of the linear transformation T with respect to the basis $\mathcal{B}$.

d.

$$\begin{aligned}
T(1) &= 1 - 0 = 1 &&= 1 \cdot 1 + 0 \cdot t + 0 \cdot t^2 \\
T(t) &= t - 1 &&= (-1) \cdot 1 + 1 \cdot t + 0 \cdot t^2 \\
T(t^2) &= t^2 - 2t &&= 0 \cdot 1 + (-2) \cdot t + 1 \cdot t^2
\end{aligned}$$

$$\begin{aligned}
[T]_{\mathcal{B}} &= \begin{bmatrix} [T(1)]_{\mathcal{B}} & [T(t)]_{\mathcal{B}} & \left[T(t^2)\right]_{\mathcal{B}} \end{bmatrix} \\
&= \begin{bmatrix} [1]_{\mathcal{B}} & [t-1]_{\mathcal{B}} & \left[t^2 - 2t\right]_{\mathcal{B}} \end{bmatrix} \\
&= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

3. Consider the vector space $V = \mathbb{P}_2$ and the basis $\mathcal{B} = \{1,\ t,\ t^2\}$ for $\mathbb{P}_2$.

(e) (8 points) Let $\mathcal{C} = \{2,\ 4t,\ 1+t^2\}$, which is also a basis for $\mathbb{P}_2$. Find the change-of-coordinates matrix $\underset{\mathcal{C}\leftarrow\mathcal{B}}{P}$.

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} [1]_{\mathcal{C}} & [t]_{\mathcal{C}} & [t^2]_{\mathcal{C}} \end{bmatrix}$$

$$\begin{aligned} 1 &= \tfrac{1}{2}\cdot 2 + 0\cdot 4t + 0\cdot(1+t^2) \\ t &= 0\cdot 2 + \tfrac{1}{4}\cdot 4t + 0\cdot(1+t^2) \\ t^2 &= \left(-\tfrac{1}{2}\right)\cdot 2 + 0\cdot 4t + 1\cdot(1+t^2) \end{aligned}$$

Therefore

$$\underset{\mathcal{C}\leftarrow\mathcal{B}}{P} = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & \frac{1}{4} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

4. Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix} \right\}$ and let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(a) (4 points) Find a basis for W.

Since $\begin{bmatrix} 1 & 2 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 1 & 7 & -1 \end{bmatrix}^T$ are not scalar multiple of each other,

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix} \right\}$$

is a basis of W.

4. Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix} \right\}$ and let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(b) (8 points) Find an **orthogonal** basis for W.

The dot product $(1,2,0) \cdot (1,7,-1) = 1 + 14 = 15 \neq 0$, so we need to use Gram-Schmidt process (without normalization). Let $u_1 = (1,2,0)$. Then

$$\begin{aligned} u_2 &= (1,7,-1) - \frac{(1,7,-1)\cdot(1,2,0)}{(1,2,0)\cdot(1,2,0)}(1,2,0) \\ &= (1,7,-1) - \frac{\cancel{15}^{\,3}}{\cancel{5}}(1,2,0) \\ &= (-2,1,-1) \end{aligned}$$

An orthogonal basis for W is

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}$$

4. Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix} \right\}$ and let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(c) (4 points) Give $\dim W$ and $\dim(W^\perp)$.

Let W be a subspace of a vector space V with dimension n. Then one can show that

$$\dim\ W + \dim\ W^\perp = n$$

(Exercise 32 of Section 6.3)

Here W is a subspace of $\mathbb{R}^3$ with $\dim\ W = 2$ (as the basis has two elements). And $3 = 2 + \dim\ W^\perp \Rightarrow \dim\ W^\perp = 1$.

4. Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix} \right\}$ and let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(d) (2 points) Which of the following geometric phrases best describes W?

a point, two points, a line, a plane, all of $\mathbb{R}^3$

(e) (2 points) Which of the following geometric phrases best describes $W^\perp$?

a point, two points, a line, a plane, all of $\mathbb{R}^3$

d. Since $\dim\ W = 2$, geometrically W is a **plane**.

e. Since $\dim\ W^\perp = 1$, geometrically W is a **line**.

4. Let $W = \text{Span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -1 \end{bmatrix} \right\}$ and let $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

(f) (8 points) Find the vector w in W which is closest to the given vector v (see above).

f. Using the orthogonal basis we found on b, we have

$$
\begin{aligned}
\text{Proj}_W v &= \frac{(1,2,3)\cdot(1,2,0)}{(1,2,0)\cdot(1,2,0)}(1,2,0) + \frac{(1,2,3)\cdot(-2,1,-1)}{(-2,1,-1)\cdot(-2,1,-1)}(-2,1,-1) \\
&= \frac{5}{5}(1,2,0) + \frac{-3}{6}(-2,1,-1) \\
&= (2, \tfrac{3}{2}, \tfrac{1}{2})
\end{aligned}
$$

5. Let $A = \begin{bmatrix} 1 & -3 & 1 & 1 & -3 \\ 2 & -1 & 5 & 1 & 2 \\ 1 & 2 & 4 & 0 & 5 \\ 3 & 1 & 9 & 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 14/5 & 2/5 & 0 \\ 0 & 1 & 3/5 & -1/5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and assume that A and B are row equivalent.

(a) (8 points) Identify a column of A which has the property that it is a linear combination of the columns of A to its left. Then give an explicit linear combination of those columns which yields your answer.

By looking at B, the first column and the second column are pivot columns, and the third column is not. Therefore, the third column has to be a linear combination of the first two columns. Then by considering vector equation and the corresponding augmented matrix

$$\begin{bmatrix} 1 \\ 5 \\ 4 \\ 9 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ -1 \\ 2 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 & -3 & 1 \\ 2 & -1 & 5 \\ 1 & 2 & 4 \\ 3 & 1 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 14/5 \\ 0 & 1 & 3/5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have that

$$\begin{bmatrix} 1 \\ 5 \\ 4 \\ 9 \end{bmatrix} = \frac{14}{5} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix} + \frac{3}{5} \begin{bmatrix} -3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$$

5. Let $A = \begin{bmatrix} 1 & -3 & 1 & 1 & -3 \\ 2 & -1 & 5 & 1 & 2 \\ 1 & 2 & 4 & 0 & 5 \\ 3 & 1 & 9 & 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 14/5 & 2/5 & 0 \\ 0 & 1 & 3/5 & -1/5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and assume that A and B are row equivalent.

(b) (6 points) Find a basis for Col A and give dim(Col A).

The set of pivot columns is a basis of $\mathrm{Col}\ A$, so

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -3 \\ -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -3 \\ 2 \\ 5 \\ 2 \end{bmatrix} \right\}$$

Therefore, $\dim(\mathrm{Col}\ A) = 3$.

5. Let $A = \begin{bmatrix} 1 & -3 & 1 & 1 & -3 \\ 2 & -1 & 5 & 1 & 2 \\ 1 & 2 & 4 & 0 & 5 \\ 3 & 1 & 9 & 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 14/5 & 2/5 & 0 \\ 0 & 1 & 3/5 & -1/5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and assume that A and B are row equivalent.

(c) (8 points) Find a basis for Nul A and give dim(Nul A).

The general solution is

$$\begin{bmatrix} -\frac{14}{5}x_3 - \frac{2}{5}x_4 \\ -\frac{3}{5}x_3 + \frac{1}{5}x_4 \\ x_3 \\ x_4 \\ 0 \end{bmatrix} = x_3 \begin{bmatrix} -\frac{14}{5} \\ -\frac{3}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -\frac{2}{5} \\ \frac{1}{5} \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Hence a basis for $\mathrm{Nul}\ A$ is

$$\left\{ \begin{bmatrix} -\frac{14}{5} \\ -\frac{3}{5} \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{2}{5} \\ \frac{1}{5} \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}. \text{ Hence } \dim(\mathrm{Nul}\ A) = 2.$$

5. Let $A = \begin{bmatrix} 1 & -3 & 1 & 1 & -3 \\ 2 & -1 & 5 & 1 & 2 \\ 1 & 2 & 4 & 0 & 5 \\ 3 & 1 & 9 & 1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 & 14/5 & 2/5 & 0 \\ 0 & 1 & 3/5 & -1/5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, and assume that A and B are row equivalent.

(d) (2 points) What is the rank of A?

$\text{Rank } A = \dim(\text{Col } A) = 3.$

(e) (3 points) TRUE or FALSE: The 4th column of A is an element of Col A.

True Any column is an element of $\text{Col } A$.

(f) (3 points) TRUE or FALSE: The 2nd row of B is an element of Row A.

True as $\text{Row } A = \text{Row } B$.

6. (a) Suppose that the characteristic polynomial of some $n \times n$ matrix A is given by

$$p_A(\lambda) = \lambda(\lambda - 5)^2(\lambda + 9)(\lambda - 3)^3.$$

i. (2 points) Determine the value of n (the number of rows/columns of A).

$$n = \text{ degree of the characteristic polynomial } = 1 + 2 + 1 + 3 = 7$$

ii. (6 points) Give all eigenvalues of A, and for each eigenvalue, give its (algebraic) multiplicity.

$$\lambda = 0 \text{ (mul. } 1), 5 \text{ (mul. } 2), -9 \text{ (mul. } 1), 3 \text{ (mul. } 3)$$

iii. (5 points) Based on the given information, can we determine if A is invertible? Explain why or why not.

Since 0 is an eigenvalue, the determinant which is the product of eigenvalues is 0. Hence A is not invertible.

iv. (5 points) Based on the given information, can we determine if A is diagonalizable? Explain why or why not.

We cannot determine because we need to check whether or not the algebraic multiplicity is equal to geometric multiplicity (or the dimension of the corresponding eigenvalue).

(b) Let $B = \begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix}$.

i. **(6 points) Find all (real or complex) eigenvalues of B.**

$$\det \begin{bmatrix} 3-\lambda & 5 \\ -5 & 3-\lambda \end{bmatrix} = (3-\lambda)^2 + 25 = \lambda^2 - 6\lambda + 34.$$

$$\lambda = \frac{6 \pm \sqrt{\cancelto{-100}{36 - 136}}}{2} = 3 \pm 5i$$

Alternatively,

$$\begin{aligned} (3-\lambda)^2 + 25 = 0 \quad &\Rightarrow \quad (\lambda - 3)^2 = -25 \quad \Rightarrow \quad (\lambda - 3)^2 = 5i \\ &\Rightarrow \quad \lambda - 3 = \pm 5i \quad \Rightarrow \quad \lambda = 3 \pm 5i \end{aligned}$$

(b) Let $B = \begin{bmatrix} 3 & 5 \\ -5 & 3 \end{bmatrix}$.

ii. **(6 points) Find an eigenvector for one of the eigenvalues of B (your choice).**

$$A - (3-5i)I = \begin{bmatrix} 5i & 5 \\ -5 & 5i \end{bmatrix} \to \begin{bmatrix} 5i & 5 \\ 0 & 0 \end{bmatrix}$$

The row reduction is immediate because the dimension of eigenspace for $3 - 5i$ is 1. (Or you can we have $i \begin{bmatrix} 5i & 5 \end{bmatrix} = \begin{bmatrix} -5 & 5i \end{bmatrix}$). Hence an eigenvector of $3 - 5i$ is

$$\begin{bmatrix} -5 \\ 5i \end{bmatrix}$$

7. (a) Let $A = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$.

i. **(10 points)** Find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$. You do not have to compute P^{-1}.

$$\det \begin{bmatrix} 2-\lambda & 6 \\ 6 & 7-\lambda \end{bmatrix} = (2-\lambda)(7-\lambda) - 36 = \lambda^2 - 9\lambda - 22 = (\lambda - 11)(\lambda + 2)$$

Therefore, the eigenvalues are $\lambda = 11, -2$.

$$\begin{aligned} A - 11I &= \begin{bmatrix} -9 & 6 \\ 6 & -4 \end{bmatrix} \to \begin{bmatrix} -3 & 2 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ A + 2I &= \begin{bmatrix} 4 & 6 \\ 6 & 9 \end{bmatrix} \to \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3 \\ -2 \end{bmatrix} \end{aligned}$$

$$A = \underbrace{\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}}_{=P} \underbrace{\begin{bmatrix} 11 & 0 \\ 0 & -2 \end{bmatrix}}_{=D} \underbrace{\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}^{-1}}_{=P^{-1}}$$

7. (a) Let $A = \begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix}$.

ii. **(5 points) If possible, find an orthogonal matrix Q and a diagonal matrix D such that $A = QDQ^T$.**

Since the two eigenvectors found in 7(a)ii are orthogonal, so we just need to normalize. Both eigenvectors have norm $\sqrt{4+9} = \sqrt{13}$.

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2/\sqrt{14} \\ 3/\sqrt{14} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 3 \\ -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 3/\sqrt{14} \\ -2/\sqrt{14} \end{bmatrix}$$

Therefore, orthogonal diagonalization is

$$\begin{bmatrix} 2 & 6 \\ 6 & 7 \end{bmatrix} = \underbrace{\begin{bmatrix} 2/\sqrt{14} & 3/\sqrt{14} \\ 3/\sqrt{14} & -2/\sqrt{14} \end{bmatrix}}_{=Q} \underbrace{\begin{bmatrix} 11 & 0 \\ 0 & -2 \end{bmatrix}}_{=D} \underbrace{\begin{bmatrix} 2/\sqrt{14} & 3/\sqrt{14} \\ 3/\sqrt{14} & -2/\sqrt{14} \end{bmatrix}}_{=Q^D}$$

(b) In this part, B denotes a 3×4 matrix, and we do not have any additional information about this matrix. For each of the following statements, answer TRUE if the statement **must** be true, answer FALSE if the statement **must** be false, and answer NOT ENOUGH INFO if there is not enough information available to determine whether the statement is true or false. You do not have to justify your answers.

i. (3 points) The columns of B are linearly dependent.

True, the columns of B are in $\mathbb{R}^3$ and there are four column vectors.

ii. (3 points) Some subset of the columns of B forms a basis for $\mathbb{R}^3$.

Not Enough Info

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The first three columns of A form a basis for $\mathbb{R}^3$ and no subset of columns of B form a basis for $\mathbb{R}^3$.

B denotes a 3×4 matrix.

iii. **(3 points)** The linear transformation $T(\mathbf{x}) = B\mathbf{x}$ maps onto $\mathbb{R}^3$.

Not Enough Info

If $B = 0$, then $T(x) = 0$ for all $x \in \mathbb{R}^3$, so cannot map onto $\mathbb{R}^3$. If $B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, then $\dim \mathrm{Nul}(B) = 1$, so $\mathrm{Rank}\ B = 3$, i.e. $\mathrm{Range}\ T = \mathbb{R}^3$. This shows that T maps onto $\mathbb{R}^3$.

iv. **(3 points)** The linear system $B\mathbf{x} = \mathbf{0}$ is consistent.

True The vector $\begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}^T$ is always a solution to above matrix equation.

B denotes a 3×4 matrix.

v. (3 points) $\dim(\operatorname{Nul} B) = 1$.

Not Enough Info

$$\operatorname{Nul}\left(\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\right) = \operatorname{Span}\left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}\right\} \Rightarrow \dim(\operatorname{Nul} B) = 1$$

$$\operatorname{Nul}\left(\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\right) = \mathbb{R}^4 \Rightarrow \dim(\operatorname{Nul} B) = 4$$