Worksheet 23 **Name:** ______________________
MATH 240 (Spring 2024) **Section:** 0111(12PM-1PM) / 0121(1PM-2PM)
May 7th, 2024 **TA:** Shin Eui Song

1. **Suggested Problems**

Problem 1 (6.6.4)**.** Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits the given data points.

$$(2,3),(3,2),(5,1),(6,0)$$

Problem 2 (7.1.11, 7.1.12)**.** Determine which of the matrices are orthogonal. If orthogonal, find the inverse.

a)

$$\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ 0 & 1/3 & -2/3 \\ 5/3 & -4/3 & -2/3 \end{bmatrix}$$

b)

$$\begin{bmatrix} .5 & .5 & -.5 & -.5 \\ .5 & .5 & .5 & .5 \\ .5 & -.5 & -.5 & .5 \\ .5 & -.5 & .5 & -.5 \end{bmatrix}$$

Problem 3 (7.1.15)**.** Orthogonally diagonalize the matrix.

$$\begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix}$$

Problem 1. The least-squares solution we need to find is

$$\underbrace{\begin{bmatrix}1 & 2\\1 & 3\\1 & 5\\1 & 6\end{bmatrix}}_{=A}\underbrace{\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}}_{=\beta}=\underbrace{\begin{bmatrix}3\\2\\1\\0\end{bmatrix}}_{=b}. \text{ Then } A^TA\beta = A^T\beta \text{ is}$$

$$\begin{bmatrix}1 & 1 & 1 & 1\\2 & 3 & 5 & 6\end{bmatrix}\begin{bmatrix}1 & 2\\1 & 3\\1 & 5\\1 & 6\end{bmatrix}\begin{bmatrix}\beta_0\\\beta_1\end{bmatrix}=\begin{bmatrix}1 & 1 & 1 & 1\\2 & 3 & 5 & 6\end{bmatrix}\begin{bmatrix}3\\2\\1\\0\end{bmatrix} \Rightarrow \begin{bmatrix}4 & 16\\16 & 74\end{bmatrix}\beta=\begin{bmatrix}6\\17\end{bmatrix}$$

Then

$$\begin{bmatrix}4 & 16 & 6\\16 & 74 & 17\end{bmatrix}\to\begin{bmatrix}1 & 4 & 1.5\\0 & 10 & -7\end{bmatrix}$$

Then we have $\beta_1 = -0.7$, and $\beta_0 + 4\beta_1 = 1.5 \Rightarrow \beta_0 = 4.3$. Therefore the least-squares line that best fits the data points is the equation

$$y = 4.3 - 0.7x$$

Problem 2.

(a) The second column of the matrix has the norm

$$\sqrt{\frac{4}{9}+\frac{1}{9}+\frac{16}{9}}=\sqrt{\frac{21}{9}}=\sqrt{\frac{7}{3}}\neq 1$$

Therefore, the set of columns of the matrix is not orthonormal. Hence, the matrix is not orthogonal.

(b) We will first prove that the set of columns of the matrix is an orthonormal basis. Note that all the entries of the matrix is either 0.5 and -0.5.

(i) The norms of the column vectors is $0.25 + 0.25 + 0.25 + 0.25 = 1$, hence the vectors are unit vectors.

(ii) For any pair of different column vectors u and v, let us write

$$u = \begin{bmatrix}u_1 & u_2 & u_3 & u_4\end{bmatrix}^T \text{ and } v = \begin{bmatrix}v_1 & v_2 & v_3 & v_4\end{bmatrix}^T.$$

Then the dot product is

$$u_1v_1 + u_2v_2 + u_3v_3 + u_4v_4$$

Then

$$u_iv_i = \begin{cases}0.25 & \text{if } u_i \text{ and } v_i \text{ has the same sign}\\-0.25 & \text{if } u_i \text{ and } v_i \text{ has the opposite sign}\end{cases}$$

By comparing signs, we always have two of u_iv_i are 0.25 and two of u_iv_i are -0.25, so the dot product $u \cdot v = 0$ whenever u and v are different.

Therefore, the columns form orthonormal basis, so the matrix is orthogonal. The inverse is given by the transpose

$$\begin{bmatrix}.5 & .5 & .5 & .5\\.5 & .5 & -.5 & -.5\\-.5 & .5 & -.5 & .5\\-.5 & .5 & .5 & -.5\end{bmatrix}$$

Problem 3. The characteristic polynomial is

$$\det \begin{bmatrix} 3-\lambda & 4 \\ 4 & 9-\lambda \end{bmatrix} = (3-\lambda)(9-\lambda) - 16 = \lambda^2 - 12\lambda + 27 - 16 = \lambda^2 - 12\lambda + 11 = (\lambda - 11)(\lambda - 1)$$

Then

$$\begin{aligned} A - I &= \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2 \\ -1 \end{bmatrix} \text{ is an eigenvector for } 1 \quad \rightsquigarrow \quad \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix} \\ A - 11I &= \begin{bmatrix} -8 & 4 \\ 4 & -2 \end{bmatrix} \to \begin{bmatrix} -2 & 1 \\ 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} \text{ is an eigenvector for } 11 \quad \rightsquigarrow \quad \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix} \end{aligned}$$

Either one can directly check that the two vectors are orthogonal by taking the dot product, or use the fact that the eigenspaces of a symmetric matrix are mutually orthogonal. Therefore

$$\begin{bmatrix} 3 & 4 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$