Two counterexamples in commutative algebra

Sam Lidz

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These are notes from my presentation in my commutative algebra class at the University of Maryland, taught by Professor Thomas Haines. They cover a large part, but not all, of what I presented.

1 An infinite dimensional Noetherian ring

This result is taken from Atiyah MacDonald section 11, exercise 4.

Let $R = k[X_1, X_2, \dots]$ be the polynomial ring in countably many variables over a field k. Let m_i be a sequence of integers such that $m_1 = 0$, $m_2 \geq 1$, and $m_{i+1} - m_i > m_i - m_{i-1}$. Let $\mathfrak{p}_i = (X_{m_i+1}, \dots, X_{m_{i+1}})$. Let $S = R \setminus \bigcup \mathfrak{p}_i$.

Theorem 1. *$S^{-1}R$ is a Noetherian ring of infinite dimension.*

It is clear that this ring has infinite dimension, since $\mathrm{ht}(S^{-1}\mathfrak{p}_i)$ grows unboundedly as i does. To show that this ring is Noetherian, we will need the following lemma.

Lemma 1. *Suppose A is a ring satisfying*

1. *$\forall \mathfrak{m} \subset A$ maximal, $A_\mathfrak{m}$ is Noetherian*
2. *$\forall x \neq 0$ in A, the set $\{\mathfrak{m} \subset A \text{ maximal} \mid x \in \mathfrak{m}\}$ is finite*

Then A is Noetherian.

Proof. Let $I \subset A$ be an ideal. I will show that I is finitely generated. Let $x_0 \in I$, let $\mathfrak{m}_1, \dots, \mathfrak{m}_r$ be the maximal ideals containing I, and let $\mathfrak{m}_{r+1}, \dots, \mathfrak{m}_{r+s}$ be the maximal ideals containing x_0 but not I. Then $\exists x_j$ $(1 \leq j \leq s)$ such that $x_j \in I$, $x_j \notin \mathfrak{m}_{r+j}$. Since each $A_{\mathfrak{m}_i}$ is Noetherian, $IA_{\mathfrak{m}_i}$ are all finitely generated, so $\exists x_{s+1}, \dots, x_t \in A$ whose images generate each of the $A_{\mathfrak{m}_i}$ $(1 \leq i \leq r)$.

Let $I_0{=}(x_0, x_1, \dots, x_t)$. Then $\forall \mathfrak{m}$ maximal in A, $I_0 A_\mathfrak{m} = IA_\mathfrak{m}$, so we have that $I_0 = I$, so I is finitely generated, so A is Noetherian. $\square$

To use this lemma, we need a characterization of the maximal ideals of $S^{-1}R$. It turns out that the $S^{-1}\mathfrak{p}_i$ are exactly the maximal ideals of $S^{-1}R$. To see this, recall that ideals of $S^{-1}R$ correspond to ideals of R contained in $\bigcup \mathfrak{p}_i$, and use the following fact.

Claim. *If $I \subseteq \bigcup \mathfrak{p}_i$ is an ideal of R, then $I \subseteq \mathfrak{p}_i$ for some i.*

Proof. If $I = (0)$, this is immediate, so assume $I \neq (0)$. Fix $N \in \mathbb{Z}_{>0}$. Note that

$$I \subseteq \bigcup_{i=1}^{\infty} \mathfrak{p}_i \subseteq \bigcup_{i=1}^{N} \mathfrak{p}_i \cup (X_{m_{N+1}+1}, X_{m_{N+1}+2}, \dots)$$

On the right of this inclusion relation, we have a finite union of prime ideals, so by prime avoidance, $I \subseteq \mathfrak{p}_i$ for some $i \leq N$, or $I \subseteq (X_{m_{N+1}+1}, X_{m_{N+1}+2}, \dots)$.

Suppose for the sake of contradiction that $I \not\subseteq \mathfrak{p}_i$ for any i. Then we have that

$$I \subseteq \bigcap_{N=1}^{\infty} (X_{m_{N+1}+1}, X_{m_{N+1}+2}, \dots) = (0)$$

But we assumed that $I \neq (0)$, so this is a contradiction, and $I \subseteq \mathfrak{p}_i$ for some i. □

Thus we have that the $S^{-1}\mathfrak{p}_i$ are exactly the maximal ideals of $S^{-1}R$, as mentioned above. Now we can prove the theorem, by checking that $S^{-1}R$ satisfies conditions 1 and 2 from the lemma.

Proof. To check condition 1, observe that

$$(S^{-1}R)_{S^{-1}\mathfrak{p}_i} = R_{\mathfrak{p}_i} \cong k(X_1, \dots, X_{m_i}, X_{m_{i+1}+1}, \dots)[X_{m_i+1}, \dots X_{m_{i+1}}]$$

is a finite polynomial ring over a field, and is thus Noetherian.

To check condition 2, it suffices to check that an element of R can be in only finitely many $\mathfrak{p}_i$, which is immediate. □

So $S^{-1}R$ is Noetherian and infinite dimensional.

2 A one dimensional ring whose polynomial ring is three dimensional

This example is taken from this math stackexchange post, with a few additional details filled in.

Let k be a field, and let A be the subring of $k(t)[[Y]]$ given by $f(0) \in k$. By $f(0)$ I mean that Y is thought of as the variable in this ring, so the elements of A are power series with constant term in k, and other coefficients in $k(t)$. Then A has a unique nonzero prime ideal, $P = \{f \mid f(0) = 0\}$. It's clear that P is the unique maximal ideal, since it contains exactly the non-unit elements of A. To see that it's the only prime, I will show that any prime ideal I containing a nonzero element must contain all non-unit elements. I would be interested to know of a more elegant proof of this fact than the one I give here.

Proof that P is the unique prime ideal in A. Suppose $a_1Y^m + a_2Y^{m+1} + \cdots \in I$. Then $a_1^{-1}Y \in A$, so $Y^{m+1} + \frac{a_2}{a_1}Y^{m+2} + \cdots \in I$. This element is $u \times Y^m$ for some $u \in A^\times$, so $Y^m \in I$, so since we assumed I is prime, we have $Y \in I$. Then, for any $a \in k(t)$, $a^2Y \in A$, so $a^2Y^2 = (aY)^2 \in I$, so by primality of I, $aY \in I$. So all elements of P are in I, so $P = I$. □

So A has dimension 1, but I will now show that $A[X]$ has dimension 3.

Proof. Consider the maps

$$A[X] \xrightarrow{\varphi} k(t)[[Y]] \xrightarrow{\psi} k(t)$$

where $\varphi(X) = t$, and $\psi(Y) = 0$. The kernel K of φ is prime, since the image of φ is an integral domain. Also, $K \subseteq P[X]$, since $P[X]$ is the kernel of $\psi \circ \varphi$. Notice also that $YX - tY \in K$, so $K \neq (0)$, and $Y \notin K$, so $K \neq P[X]$. So we have a chain of prime ideals $(0) \subsetneq K \subsetneq P[X] \subsetneq P[X] + (X)$ of length 3. So the dimension of $A[X]$ is 3 (see Atiyah Macdonald section 11 exercise 6). $\square$