Stat 400 Test 2 Solutions

1. Since $f(x) = c(1+x)^{-3},\ x \geq 0$ for some constant c, we have

$$1 = \int_0^\infty \frac{c}{(1+x)^3}\, dx \;=\; -\frac{c}{2(1+x)^2}\Big|_0^\infty = \frac{c}{2}$$

So $c = 2$,, and for $x \geq 0$, $F_X(x) = \int_0^x \frac{2}{(1+y)^3}\, dy = 1 - \frac{1}{(1+x)^2}$. (a) **Ans.** $F_X(1) + 1 - F_X(3) = 1 - \frac{1}{4} + \frac{1}{16} = 0.8125$. (b) Here $Y = g(X) = (1+X)^2$, so $g^{-1}(y) = \sqrt{y} - 1,\ y \geq 1$. Then $F_Y(y) = F_X(\sqrt{y} - 1) = 1 - y^{-1},\ y \geq 1$, and $f_Y(y) = y^{-2},\ y \geq 1$.

2. The total roundoff error is $\sum_{i=1}^{1000} V_i$, since if each unrounded number is called Y_i, then by definition the rounded number is $Y_i + V_i$, and the difference between the total of the rounded numbers and the total of the unrounded numbers is $\sum_{i=1}^{1000}(Y_i + V_i) - \sum_{i=1}^{1000} Y_i = \sum_{i=1}^{1000} V_i$. Now $V_i \sim$ Unif$[-.5, .5]$ implies that $\mu_V = EV_i = 0$, $\sigma_V^2 = \text{Var}(V_i) = \frac{1}{12}$, and

$$P(|\sum_{i=1}^{1000} V_i| \leq 20) \;=\; P(-.02 \leq \frac{1}{1000}\sum_{i=1}^{1000} V_i \leq .02)$$

which by the Central Limit Theorem is $\approx \Phi\Big(\frac{.02}{\sqrt{1/12000}}\Big) - \Phi\Big(\frac{-.02}{\sqrt{1/12000}}\Big) = 2\Phi(2.19) - 1 = .9714$.

3. (a) Since M is the sum of 4800 *iid* coin-toss random variables (with values 0 or 1), each with the same probability $\int_0^{.25} du = 0.25$ of value 1, it is a Binom(4800, .25) r.v.
(b). By the normal approximation to the binomial, $P(M \leq 1280) \approx \Phi((1280 - 4800(.25) + .5)/\sqrt{4800(.25)(.75)}) = \Phi(80.5/30) = 0.9963$.
(c) By Law of Large Numbers, $\frac{1}{4800}\sum_{i=1}^{4800} U_i^4 \approx E(U_1^4) = \int_0^1 u^4\, du = 1/5$.

4. (a). Direct combination of joint pmf values $p_{X,Y}(x,y)$ with $x > y$ gives answwer 0.56. (b) Marginal pmf's are given by $p_Y(-3) = .2 = p_Y(-1)$, $p_Y(0) = .3 = p_Y(4)$, $p_X(-1) = p_X(2) = .3$, $p_X(1) = .4$, and $E(Y) = (0+4)(.3) + (-3-1)(.2) = 0.4$. (c) **Not** independent, e.g. because $p_{X,Y}(2,-3) = .05 \neq .06 = p_X(2)p_Y(-3)$.

5. Recall for any r.v. X that $E(X^2) = \mu_X^2 + \sigma_X^2$. Here $X = V + 2W$, $\mu_X = 2 + 2(2) = 6$, $\sigma_X^2 = 5 + (2^2)5 = 25$, so $EX^2 = 6^2 + 25 = 61$.